

# $\theta_C$ from the Dihedral Flavor Symmetries $D_7$ and $D_{14}$

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## Abstract

In [1] it has been shown that the Cabibbo angle  $\theta_C$  might arise from a dihedral flavor symmetry which is broken to different (directions of) subgroups in the up and the down quark sector. This leads to a prediction of  $\theta_C$  in terms of group theoretical quantities only, i.e. the index  $n$  of the dihedral group  $D_n$ , the index  $j$  of the fermion representation  $\mathbf{2}_j$  and the preserved subgroups indicated by  $m_u$  and  $m_d$ . Here we construct a low energy model which incorporates this idea. The gauge group is the one of the Standard Model and  $D_7 \times Z_2^{(aux)}$  serves as flavor symmetry. The additional  $Z_2^{(aux)}$  is necessary in order to maintain two sets of Higgs fields, one which couples only to up quarks and another one coupling only to down quarks. We assume that  $D_7$  is broken spontaneously at the electroweak scale by vacuum expectation values of  $SU(2)_L$  doublet Higgs fields. The quark masses and mixing parameters can be accommodated well. Furthermore, the potential of the Higgs fields is studied numerically in order to show that the required configuration of the vacuum expectation values can be achieved. We also comment on more minimalist models which explain the Cabibbo angle in terms of group theoretical quantities, while  $\theta_{13}^q$  and  $\theta_{23}^q$  vanish at leading order. Finally, we perform a detailed numerical study of the lepton mixing matrix  $V_{MNS}$  in which one of its elements is entirely determined by the group theory of a dihedral symmetry. Thereby, we show that nearly tri-bi-maximal mixing can also be produced by a dihedral flavor group with preserved subgroups.

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# 1 Introduction

Discrete groups have been widely used as flavor symmetry. However, only in some special cases there is a direct connection between the flavor group  $G_F$  and the resulting mixing pattern for the fermions, i.e. a correlation which does not rely on further parameter equalities not induced by  $G_F$ . Such cases occur in the  $A_4$  ( $T'$ ) models [2, 3] as well as in our systematic study of the dihedral groups [1] where the key feature is the fact that  $G_F$  is broken in a non-trivial way, i.e. one has to demand that certain of its subgroups are preserved in different sectors of the theory. Especially, the fact that sizable mixing results from the mismatch of two different (directions of) subgroups is used in the  $A_4$  ( $T'$ ) models as well as in an application of the group  $D_7$  shown in [1]. In the group  $A_4$  ( $T'$ ) which has been studied in great detail tri-bi-maximal mixing (TBM) in the lepton sector is predicted, if one assumes that the left-handed leptons transform as a triplet under  $A_4$  ( $T'$ ), and the left-handed conjugate leptons,  $e^c$ ,  $\mu^c$  and  $\tau^c$ , as the three non-equivalent one-dimensional representations of the group. There exist two sets of gauge singlets which transform non-trivially under  $A_4$  ( $T'$ ): one set only couples to neutrinos at the leading order (LO), while the other one only to charged leptons (fermions). The first one breaks  $A_4$  ( $T'$ ) spontaneously down to  $Z_2$  ( $Z_4$ ) and the latter one down to  $Z_3$ . The lepton mixing then stems from two sectors in which different subgroups of  $A_4$  ( $T'$ ) are conserved. In contrast to this, the up quark and down quark mass matrix preserve the same subgroup at LO. Similarly, it has recently been shown that such a mechanism can also be implemented with other discrete groups, for example the dihedral groups  $D_n$  and  $D'_n$ . In a first application we observed in [1] that the Cabibbo angle  $\theta_C$  or equivalently the CKM matrix element  $|V_{us}|$  can be predicted in terms of group theoretical indices only, such as the index  $n$  of the group  $D_n$ , the index  $j$  of the representation under which the (left-handed) quarks transform and the misalignment of the two different (directions of) subgroups  $Z_2 = \langle BA^{m_u} \rangle$  and  $Z_2 = \langle BA^{m_d} \rangle$ :

$$|V_{us}| = \left| \cos \left( \frac{\pi (m_u - m_d) j}{n} \right) \right| \quad (1)$$

There is a crucial difference between the models using a dihedral symmetry and  $A_4$  ( $T'$ ) as flavor symmetry, namely the issue whether the representations under which the Higgs (flavon) fields transform are chosen or not. In our study on dihedral symmetries [1] we always assumed that for each representation  $\mu$  which (has a component which) transforms trivially under the relevant subgroup there exist(s) (a) Higgs field(s) transforming as  $\mu$  and acquiring a non-vanishing vacuum expectation value (VEV). Due to this the resulting mass matrices are only determined by the choice of the fermion representations, the dihedral group and the preserved subgroups, but not by the choice of the scalar fields. This makes our results less arbitrary. However, in the case of the  $A_4$  ( $T'$ ) it is necessary to choose the transformation properties of the scalar fields properly, i.e. one has to exclude scalars which transform as non-trivial singlets under  $A_4$  ( $T'$ ) and couple to neutrinos at LO, in order to arrive at the TBM scenario [3, 4].

In this paper we investigate the idea of [1] by constructing a viable (low energy) model for the quark sector. The gauge group is chosen to be the one of the Standard Model (SM). We study the mass matrices numerically in order to demonstrate that all quark masses and mixing parameters can be accommodated. We discuss the Higgs potential under the assumption that all involved fields are copies of the SM Higgs doublet. Furthermore, instead of accommodating all quark mixing angles at LO it is also worth studying setups in which the Cabibbo angle is predicted in terms of group theoretical quantities, while the two other mixing angles are zero. This can be done in at least two ways: *a.*) one can choose the representations under which the scalar fields should transform or *b.*) one can look at cases in which the preserved subgroup in each

sector is not only a  $Z_2$ , but rather is itself a dihedral group  $D_q$ ,  $q > 1$ . Finally, we motivate possible extensions of the model to the lepton sector by performing a detailed numerical study. Additionally, we show that nearly TBM can be also accommodated by using a dihedral flavor symmetry.

The paper is organized as follows: in Section 2 we review the findings of [1] which we explore in more detail; Section 3 treats the mixing matrix  $V_{CKM}$  only - in an analytic way as well as numerically; in Section 4 we study a model for the quark sector which incorporates the idea presented in [1] and show that it accommodates both quark mixings and masses; in Section 5 the Higgs potential, belonging to one of the models of Section 4, is discussed and a numerical analysis proves that the advocated VEV structure can be achieved. Section 6 is devoted to ansätze in which only the Cabibbo angle is generated at LO. In Section 7 we perform the same analysis, as for the quark mixing matrix  $V_{CKM}$  in Section 3, for the lepton mixing matrix  $V_{MNS}$  in order to see whether the fact that one element of the mixing matrix is described only in terms of group theoretical quantities is also applicable here. Thereby, we assume that the neutrinos are Dirac particles as all the other fermions and possess the same mass ordering, i.e. that they are normally ordered. Finally, we summarize our results in Section 8. Appendix A contains the possible forms of the mixing matrices  $V_{CKM}$  and  $V_{MNS}$ , in Appendix B the group theory of  $D_7$ , i.e. the flavor group used in Section 4 and Section 5, is presented. Further details of the study of the Higgs sector are delegated to Appendix C.

## 2 Basics

In this section we repeat the findings of [1] concerning the possible structure of (Dirac) mass matrices with a non-vanishing determinant. They are of the form:

$$M_1 = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}, \quad M_2 = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & C & 0 \end{pmatrix} \quad (2)$$

$$M_3 = \begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & D & E \end{pmatrix}, \quad (3)$$

$$M_4 = \begin{pmatrix} 0 & A & B \\ C & D & E \\ -C e^{-i\phi j} & D e^{-i\phi j} & E e^{-i\phi j} \end{pmatrix} \quad \text{and} \quad M_5 = \begin{pmatrix} A & C & C e^{-i\phi k} \\ B & D & E \\ B e^{-i\phi j} & E e^{-i\phi(j-k)} & D e^{-i\phi(j+k)} \end{pmatrix} \quad (4)$$

where  $A, B, C, D, E$  are complex numbers which are products of Yukawa couplings and VEVs,  $\phi = \frac{2\pi}{n} m$  ( $n$ : index of the dihedral group,  $m$ : index of the breaking direction) and  $j, k$  are indices of representations. Regarding  $M_4$  notice that we presented in [1] the transpose of this matrix. However, a transposition in general only corresponds to the exchange of the transformation properties of the left-handed and left-handed conjugate fields under the flavor symmetry and therefore does not change the group theoretical part of the discussion about the preserved subgroups. Furthermore, these matrices are determined up to permutations of columns and rows which also only correspond to permutations among the three generations of the fields. As in [1] we work in the SM and with the assumption that all Higgs fields  $H$  in the model are copies of the SM one. Therefore the displayed mass matrices are those for down-type fermions, i.e. down quarks and charged leptons. The corresponding ones for up-type fermions, i.e. up quarks and (Dirac) neutrinos, require some changes due to the fact that only the conjugates of the Higgs fields,  $\epsilon H^*$ , couple to up-type fermions and we use complex matrices for the two-dimensional

representations of  $D_n$ . According to the rules of [1] on how to deduce the up-type fermion mass matrices from the shown ones,  $M_4$  and  $M_5$  are of the form

$$M_4 = \begin{pmatrix} 0 & A & B \\ C e^{i\phi j} & D e^{i\phi j} & E e^{i\phi j} \\ -C & D & E \end{pmatrix} \quad \text{and} \quad M_5 = \begin{pmatrix} A & C e^{i\phi k} & C \\ B e^{i\phi j} & D e^{i\phi(j+k)} & E e^{i\phi(j-k)} \\ B & E & D \end{pmatrix} \quad (5)$$

An explicit example is given in Section 4, where a model for quark masses and mixings is presented.

We concentrate on the last two forms,  $M_4$  and  $M_5$ . This we do for two reasons: first we want to accommodate all masses and mixing parameters at tree level in the first part of the work, i.e. we do not want to rely on the fact that one mixing angle is only generated by higher order effects; second we would like to have the same mass matrix structure for up quarks (Dirac neutrinos) and down quarks (charged leptons).

Let us briefly mention the origin of the matrix structures  $M_4$  and  $M_5$ . The flavor symmetry is a single-valued dihedral group  $D_n$  with arbitrary index  $n$ . The preserved subgroup is in both cases  $Z_2 = \langle BA^m \rangle$  where  $m = 0, 1, \dots, n-1$ . This subgroup allows non-vanishing VEVs for the following one-dimensional representations:  $\underline{\mathbf{1}}_1$  (is always allowed to have a VEV),  $\underline{\mathbf{1}}_3$  for  $m$  even and  $\underline{\mathbf{1}}_4$  for  $m$  odd. All two-dimensional representations acquire a so-called structured VEV, i.e. for two fields  $\psi_{1,2}$  transforming as an irreducible two-dimensional representation  $\underline{\mathbf{2}}_p$  their VEVs have to have the correlation:  $\langle \psi_1 \rangle = \langle \psi_2 \rangle e^{-\frac{2\pi i p m}{n}}$ . The notation of the representations used here is according to the one given in [1].

In case of  $M_4$  we take the left-handed fields  $L$  to transform as  $\underline{\mathbf{1}}_k + \underline{\mathbf{2}}_j$  under the dihedral group, and the left-handed conjugate fields  $L^c$  transform as the three singlets  $\underline{\mathbf{1}}_{i_1} + \underline{\mathbf{1}}_{i_2} + \underline{\mathbf{1}}_{i_3}$ . A complete study of all possible assignments shows that one of the entries in the first row needs to be zero in order to prevent the determinant of the matrix from being zero. The matrix structure  $M_5$  arises, if both left-handed and left-handed conjugate fermions transform as  $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ ,  $L \sim (\underline{\mathbf{1}}_i, \underline{\mathbf{2}}_j)$  and  $L^c \sim (\underline{\mathbf{1}}_l, \underline{\mathbf{2}}_k)$ . Here the constraint  $\det(M) \neq 0$  enforces the (11) element of the mass matrix to be non-zero, i.e.  $\underline{\mathbf{1}}_i \times \underline{\mathbf{1}}_l$  has to have a non-vanishing VEV. Note that the indices of the representations do not need to coincide, i.e.  $i \neq l$  and  $j \neq k$  is possible, although it might not be favorable from the viewpoint of a (partially) unified model.

To study the mixing matrices arising from  $M_4$  and  $M_5$  for down-type as well as up-type fermions we observe that the products  $M_i M_i^\dagger$ ,  $i = 4, 5$ , can be written in the general form

$$\begin{pmatrix} a & b e^{i\beta} & b e^{i(\beta+\phi j)} \\ b e^{-i\beta} & c & d e^{i\phi j} \\ b e^{-i(\beta+\phi j)} & d e^{-i\phi j} & c \end{pmatrix}$$

where  $a, b, c, d$  and  $\beta$  are real functions of  $A, B, C, D$  and  $E$ . The phase  $\beta$  lies in the interval  $[0, 2\pi)$ . Since we work in the basis in which the left-handed fields are on the left-hand side and the left-handed conjugate fields on the right-hand side, the unitary matrix which diagonalizes  $M_i M_i^\dagger$  acts on the left-handed fields and therefore determines the physical mixing matrices, i.e. the CKM matrix and the MNS matrix. The three eigenvalues are given as  $(c-d)$ ,  $\frac{1}{2}(a+c+d - \sqrt{(a-c-d)^2 + 8b^2})$  and  $\frac{1}{2}(a+c+d + \sqrt{(a-c-d)^2 + 8b^2})$ . Assuming this ordering of the eigenvalues the mixing matrix  $U$  which fulfills  $U^\dagger M_i M_i^\dagger U = \text{diag}$  is of the form:

$$U = \begin{pmatrix} 0 & \cos(\theta) e^{i\beta} & \sin(\theta) e^{i\beta} \\ -\frac{1}{\sqrt{2}} e^{i\phi j} & -\frac{\sin(\theta)}{\sqrt{2}} & \frac{\cos(\theta)}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{\sin(\theta)}{\sqrt{2}} e^{-i\phi j} & \frac{\cos(\theta)}{\sqrt{2}} e^{-i\phi j} \end{pmatrix}$$

The angle  $\theta$  is determined to be  $\tan(2\theta) = \frac{2\sqrt{2}b}{c+d-a}$ . Therefore it lies in the interval  $[0, \frac{\pi}{2})$ . If the three eigenvalues are not degenerate, the eigenvectors are determined by them up to phases <sup>1</sup>. Therefore the variants of the mixing matrix  $U$  are given by permutations of the columns. With this at hand we can look for possible interesting structures in the mixing matrix which is just the product of two matrices of this form, as we assume that the up quark (Dirac neutrino) and the down quark (charged lepton) mass matrix is either of the form  $M_4$  or  $M_5$ . The mixing matrix is then of the form  $V = W_1^T W_2^*$  with  $W_i$  being a variant of the matrix  $U$  above. For  $V = V_{CKM}$  we have  $W_1 \equiv U_u$  which is the unitary matrix diagonalizing the up quark mass matrix and  $W_2 \equiv U_d$  which is the corresponding matrix for the down quarks. In case of  $V = V_{MNS}$ ,  $W_1$  is equivalent to  $U_l$  and  $W_2$  to  $U_\nu$  <sup>2</sup>. The matrix  $W_i$  contains the group theoretical phase  $\phi_i$  according to the breaking direction  $m_i$ , the angle  $\theta_i$  and the phase  $\beta_i$ . For  $W_1 \equiv U_u$  we also use the notation  $\phi_u$ ,  $m_u$ ,  $\theta_u$  and  $\beta_u$ . An analogous convention is used for  $U_d$ ,  $U_l$  and  $U_\nu$ . It turns out that one of the elements is determined by the index  $j$  of the representation  $\mathbf{2}_j$  under which two of the left-handed fields transform and the difference of the group theoretical phases  $\phi_1$  and  $\phi_2$  only. These phases do not depend on the values of the parameters  $A, B, \dots$  (and therefore also not on  $a, b, c, d$  and  $\beta$ ), but only on the index  $n$  of the group  $D_n$  and the indices  $m_1$  and  $m_2$  being the parameters that determine the subgroup to which the Higgs fields break  $D_n$  down. Therefore this element is determined by fundamental values of the model only and not by an arbitrarily tunable number. The actual form of (the absolute value of) the element is

$$\frac{1}{2} \left| 1 + e^{i(\phi_1 - \phi_2)j} \right| = \left| \cos\left((\phi_1 - \phi_2) \frac{j}{2}\right) \right| = \left| \cos\left(\frac{\pi}{n} (m_1 - m_2)j\right) \right| \quad (6)$$

Note that this value is only non-trivial, if  $m_1 \neq m_2$ , i.e. the (directions of the) subgroups which are preserved in the up quark (Dirac neutrino) sector and the down quark (charged lepton) sector are not the same, i.e. only their mismatch leads to non-trivial mixing. This element can be traced back to the eigenvectors which correspond to the eigenvalue  $c - d$  in the up quark (Dirac neutrino) and the down quark (charged lepton) sector, i.e. the product of these two eigenvectors gives rise to the element  $\frac{1}{2} (1 + e^{i(\phi_1 - \phi_2)j})$ . Therefore the ordering of the eigenvectors in the up quark (Dirac neutrino) and down quark (charged lepton) sector determines in which position of the mixing matrix the fixed element appears. As  $V_{CKM} = U_u^T U_d^*$ , the fact whether the eigenvalue  $c - d$  is associated with up, charm or top quark mass determines the row in which the element appears while the choice of the eigenvalue  $c - d$  to be either  $m_d$ ,  $m_s$  or  $m_b$  determines the column. Analogously, the choice whether  $m_e$ ,  $m_\mu$  or  $m_\tau$  is given by  $c - d$  determines the row, while the column in which the element appears is given by the fact which of the (Dirac) neutrino masses,  $m_1$ ,  $m_2$  or  $m_3$ , is equal to  $c - d$ .

In [1] we already mentioned that we can accommodate the CKM matrix element  $|V_{us}|$  by  $\cos(\frac{3\pi}{7}) \approx 0.2225$ , i.e. by taking  $n = 7$  and for example  $j = 3$ ,  $m_u = 1$  and  $m_d = 0$ . Here we show first which of the other elements of  $V_{CKM}$  can also be accommodated well by the form  $\left| \cos\left(\frac{\pi}{n} (m_u - m_d)j\right) \right|$ .

The elements of  $V_{CKM}$  are precisely measured [5]

$$|V_{CKM}| = \begin{pmatrix} 0.97383_{-0.00023}^{+0.00024} & 0.2272_{-0.0010}^{+0.0010} & (3.96_{-0.09}^{+0.09}) \times 10^{-3} \\ 0.2271_{-0.0010}^{+0.0010} & 0.97296_{-0.00024}^{+0.00024} & (42.21_{-0.80}^{+0.10}) \times 10^{-3} \\ (8.14_{-0.64}^{+0.32}) \times 10^{-3} & (41.61_{-0.78}^{+0.12}) \times 10^{-3} & 0.999100_{-0.000004}^{+0.000034} \end{pmatrix}$$

<sup>1</sup>Since the eigenvectors should be normalized their length is fixed to one.

<sup>2</sup>Throughout the paper we assume that the neutrinos are Dirac particles for simplicity. Therefore  $V_{MNS}$  has the same structure as  $V_{CKM}$ , i.e. there are no (additional) Majorana phases present in the lepton sector.

together with the Jarlskog invariant [6]  $J_{CP} = (3.08^{+0.16}_{-0.18}) \times 10^{-5}$ . We restrict ourselves to values of  $n$  smaller than 30, since the index  $n$  of the dihedral group  $D_n$  is correlated to its order and larger values of  $n$  correspond to larger groups. Enforcing  $n < 30$  leads to a group order smaller than 60 which seems to be reasonable. Then we see that we can put the elements of the  $1 - 2$  sub-block, i.e.  $|V_{ud}|$ ,  $|V_{us}|$ ,  $|V_{cd}|$  and  $|V_{cs}|$ , into the form  $|\cos(\frac{\pi}{n}(m_u - m_d)j)|$ . As  $|V_{cd}| \approx |V_{us}|$  holds to good accuracy, also  $|V_{cd}|$  can be described well by  $\cos(\frac{3\pi}{7})$ . Furthermore  $|V_{ud}| \approx |V_{cs}|$  can be approximated well as  $\cos(\frac{\pi}{14}) \approx 0.9749$  which points towards the flavor group  $D_{14}$ . Note that the value of  $|V_{ud}|$  as well as of  $|V_{cs}|$  can be accommodated even a bit better with  $\cos(\frac{2\pi}{27}) \approx 0.9730$ . However, we do not use this, as it needs the group  $D_{27}$  which is a group of order 54 and therefore already quite large. Note that, even if  $|V_{us}|$  is taken to be  $\cos(\frac{3\pi}{7})$ , there is no unique solution which flavor symmetry has to be used and to which subgroup it has to be broken, since for example taking  $j = 1$ ,  $m_u = 3$ ,  $m_d = 0$  and  $n = 7$  leads to  $|\cos(\frac{\pi}{n}(m_u - m_d)j)| = |\cos(\frac{3\pi}{7})|$  as well as  $j = 3$ ,  $m_u = 1$ ,  $m_d = 0$  and  $n = 7$  and also  $j = 1$ ,  $m_u = 6$ ,  $m_d = 0$  and  $n = 14$ . As  $|\cos(\frac{4\pi}{7})|$  equals  $|\cos(\frac{3\pi}{7})|$ , this allows us to deduce further possible values for  $j$ ,  $m_u$ ,  $m_d$  and  $n$  like  $j = 1$ ,  $m_u = 0$ ,  $m_d = 4$  and  $n = 7$ .

In the next section we study the cases  $|V_{us}|$  and  $|V_{cd}|$  equal to  $\cos(\frac{3\pi}{7})$  and  $|V_{ud}|$  and  $|V_{cs}|$  equal to  $\cos(\frac{\pi}{14})$  in greater detail and thereby check whether we can always adjust the two other mixing angles  $\theta_{13}^q$  and  $\theta_{23}^q$  with the free angles  $\theta_u$  and  $\theta_d$  and also the Jarlskog invariant  $J_{CP}$  with the difference of the two phases  $\beta_u$  and  $\beta_d$ .

### 3 Analysis of $V_{CKM}$ only

#### 3.1 Remarks

There are six possible forms for  $U$  which correspond to different identifications of the eigenvalues. However, the fact that  $m_u \ll m_c \ll m_t$  and  $m_d \ll m_s \ll m_b$  allows only three of them, as the eigenvalue  $\frac{1}{2}(a + c + d - \sqrt{(a - c - d)^2 + 8b^2})$  is smaller than  $\frac{1}{2}(a + c + d + \sqrt{(a - c - d)^2 + 8b^2})$ . Therefore, we will only vary the position of the eigenvector belonging to the eigenvalue  $c - d$ , while keeping the ordering of the two others fixed. The three different forms of the mixing matrix  $U$  are then:

$$\begin{aligned}
U &= \begin{pmatrix} 0 & \cos(\theta) e^{i\beta} & \sin(\theta) e^{i\beta} \\ -\frac{1}{\sqrt{2}} e^{i\phi j} & -\frac{\sin(\theta)}{\sqrt{2}} & \frac{\cos(\theta)}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{\sin(\theta)}{\sqrt{2}} e^{-i\phi j} & \frac{\cos(\theta)}{\sqrt{2}} e^{-i\phi j} \end{pmatrix} \\
U' &= \begin{pmatrix} \cos(\theta) e^{i\beta} & 0 & \sin(\theta) e^{i\beta} \\ -\frac{\sin(\theta)}{\sqrt{2}} & -\frac{1}{\sqrt{2}} e^{i\phi j} & \frac{\cos(\theta)}{\sqrt{2}} \\ -\frac{\sin(\theta)}{\sqrt{2}} e^{-i\phi j} & \frac{1}{\sqrt{2}} & \frac{\cos(\theta)}{\sqrt{2}} e^{-i\phi j} \end{pmatrix} \\
U'' &= \begin{pmatrix} \cos(\theta) e^{i\beta} & \sin(\theta) e^{i\beta} & 0 \\ -\frac{\sin(\theta)}{\sqrt{2}} & \frac{\cos(\theta)}{\sqrt{2}} & -\frac{1}{\sqrt{2}} e^{i\phi j} \\ -\frac{\sin(\theta)}{\sqrt{2}} e^{-i\phi j} & \frac{\cos(\theta)}{\sqrt{2}} e^{-i\phi j} & \frac{1}{\sqrt{2}} \end{pmatrix}
\end{aligned}$$

Combining them leads to nine distinct possibilities for the CKM matrix whose forms are displayed in Appendix A. Since we already mentioned that we want to concentrate on the  $1 - 2$  sub-block we only need to consider the four possible combinations which involve the matrices  $U$  and  $U'$  as mixing matrices.

### 3.2 Numerical Study

In this section we discuss the results of our fits to the CKM matrix where we assume that one of the matrix elements in the 1 – 2 sub-block is determined by group theory, as explained in the preceding section. There are three free parameters in the mixing matrix:  $\theta_{u,d}$  and  $\alpha = \beta_u - \beta_d$ . We use these to fit the other two mixing angles  $\theta_{13}^q$  and  $\theta_{23}^q$  as well as the CP violation  $J_{CP}$ . The forms of  $V_{mix}$  presented in Appendix A show that two of the elements  $|V_{ub}|$ ,  $|V_{cb}|$ ,  $|V_{td}|$  and  $|V_{ts}|$  are determined by  $\cos(\theta_{u,d})$  in each of the four different cases. As these elements are small, the free angles  $\theta_u$  and  $\theta_d$  are restricted to be  $\theta_{d,u} \approx \frac{\pi}{2}$ . Therefore  $\theta_{d,u}$  is expanded around  $\frac{\pi}{2}$ ,  $\theta_{d,u} = \frac{\pi}{2} - \epsilon_{d,u}$ ,  $\epsilon_{d,u} > 0$ . The resulting four CKM matrices are (up to the first order in  $\epsilon_{u,d}$ )

$$|V_{CKM}^{11}| \approx \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) & \cos(\frac{3\pi}{7})\epsilon_d \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) & \frac{1}{2} |(1 + e^{\frac{\pi}{7}i})\epsilon_d - 2e^{i\alpha}\epsilon_u| \\ \cos(\frac{3\pi}{7})\epsilon_u & \frac{1}{2} |(1 + e^{\frac{\pi}{7}i})\epsilon_u - 2e^{i\alpha}\epsilon_d| & 1 \end{pmatrix} \quad (7)$$

$$|V_{CKM}^{12}| \approx \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14})\epsilon_d \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) & \frac{1}{2} |(1 + e^{\frac{6\pi}{7}i})\epsilon_d - 2e^{i\alpha}\epsilon_u| \\ \frac{1}{2} |(1 + e^{\frac{6\pi}{7}i})\epsilon_u - 2e^{i\alpha}\epsilon_d| & \cos(\frac{\pi}{14})\epsilon_u & 1 \end{pmatrix} \quad (8)$$

$$|V_{CKM}^{21}| \approx \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) & \frac{1}{2} |(1 + e^{\frac{6\pi}{7}i})\epsilon_d - 2e^{i\alpha}\epsilon_u| \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) & \cos(\frac{\pi}{14})\epsilon_d \\ \cos(\frac{\pi}{14})\epsilon_u & \frac{1}{2} |(1 + e^{\frac{6\pi}{7}i})\epsilon_u - 2e^{i\alpha}\epsilon_d| & 1 \end{pmatrix} \quad (9)$$

$$|V_{CKM}^{22}| \approx \begin{pmatrix} \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7}) & \frac{1}{2} |(1 + e^{\frac{\pi}{7}i})\epsilon_d - 2e^{i\alpha}\epsilon_u| \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) & \cos(\frac{3\pi}{7})\epsilon_d \\ \frac{1}{2} |(1 + e^{\frac{\pi}{7}i})\epsilon_u - 2e^{i\alpha}\epsilon_d| & \cos(\frac{3\pi}{7})\epsilon_u & 1 \end{pmatrix} \quad (10)$$

Without loss of generality we have set the representation index  $j$  to 1, the group theoretical phase  $\phi_u$  to zero ( $m_u = 0$ ) and the phase  $\phi_d$  to  $\frac{2\pi}{14}$  ( $m_d = 1$ ,  $n = 14$ ) for Eq.(7) and Eq.(10), while we take it to be  $\frac{6\pi}{7}$  ( $m_d = 3$ ,  $n = 7$ ) for Eq.(8) and Eq.(9).

Comparing Eq.(7) to the best fit values of  $|V_{ub}|$  and  $|V_{td}|$  given in [5] leads to  $\epsilon_u \approx 0.0366$  and  $\epsilon_d \approx 0.0178$ . The phase  $\alpha$  is then mainly determined by the values of  $|V_{cb}|$  and  $|V_{ts}|$ . A numerical computation leads to a best fit for  $\alpha \approx 4.810$ <sup>3</sup>. Furthermore one can calculate  $J_{CP}$  in this case:

$$\begin{aligned} J_{CP}^{11} &= \frac{1}{8} \sin\left(\frac{\pi}{7}\right) \sin\left(\frac{\pi}{14}\right) \sin(2\theta_d) \sin(2\theta_u) \sin\left(\frac{\pi}{14} - \alpha\right) \\ &\approx \frac{1}{2} \sin\left(\frac{\pi}{7}\right) \sin\left(\frac{\pi}{14}\right) \sin\left(\frac{\pi}{14} - \alpha\right) \epsilon_u \epsilon_d \end{aligned}$$

A similar analysis can be carried out for the three other matrices  $V_{CKM}^{12}$ ,  $V_{CKM}^{21}$  and  $V_{CKM}^{22}$  with similar results which we have collected in Table 1. The value of  $J_{CP}$  belonging to  $V_{CKM}^{22}$ , i.e.  $J_{CP}^{22}$ , is of the same form as  $J_{CP}^{11}$ . For  $V_{CKM}^{12}$  and  $V_{CKM}^{21}$  one finds

$$\begin{aligned} J_{CP}^{12} = J_{CP}^{21} &= -\frac{1}{8} \sin\left(\frac{6\pi}{7}\right) \sin\left(\frac{3\pi}{7}\right) \sin(2\theta_d) \sin(2\theta_u) \sin\left(\frac{3\pi}{7} - \alpha\right) \\ &\approx -\frac{1}{2} \sin\left(\frac{6\pi}{7}\right) \sin\left(\frac{3\pi}{7}\right) \sin\left(\frac{3\pi}{7} - \alpha\right) \epsilon_u \epsilon_d \end{aligned}$$

As one can see in Table 1,  $\epsilon_{u,d}$  have to be larger in case of  $V_{CKM}^{22}$ , since they are determined by  $|V_{cb}|$  and  $|V_{ts}|$ . In this way the expansion of  $\theta_{u,d}$  around  $\frac{\pi}{2}$  gets worse and the second order in  $\epsilon_{u,d}$  becomes important. This can be seen best in  $|V_{us}| \approx 0.2225$  and  $|V_{cd}| \approx 0.2225$  which

<sup>3</sup>We performed a  $\chi^2$  fit of  $J_{CP}$  and all elements of  $|V_{CKM}|$  excluding the one which is fixed by group theory. Instead of taking the (very small) experimental errors we simply assumed 10% errors for all quantities.

Parameters	$V_{CKM}^{11}$	$V_{CKM}^{12}$	$V_{CKM}^{21}$	$V_{CKM}^{22}$
$\epsilon_u$	+0.0364	+0.0427	+0.00831	+0.188
$\epsilon_d$	+0.0177	+0.00405	+0.0433	+0.191
$\alpha$	4.810	2.355	1.764	0.2056

Table 1: Fit results for  $\epsilon_{u,d}$  ( $\theta_{u,d}$ ) and the phase  $\alpha$  for  $V_{CKM}$  with either  $|V_{ud}|$ ,  $|V_{us}|$ ,  $|V_{cd}|$  or  $|V_{cs}|$  being group theoretically determined.

are lowered to 0.2186(5) such that the discrepancy between the experimentally measured value and the result of the fit gets larger. However, corrections from higher-dimensional operators and explicit breakings of the residual subgroups can lead to further contributions allowing all data to be fitted successfully.

## 4 Analysis of the Quark Sector

After having shown that one element of  $V_{CKM}$  can be explained in terms of group theoretical indices only and studying this issue numerically we want to go a step further and construct a viable model at least for the quark sector which includes this issue. The model is viable, if we find a numerical solution which accommodates not only the mixing parameters contained in  $V_{CKM}$ , but also the quark masses. Due to the strong hierarchy among the quarks this is a non-trivial task, although the number of parameters in the mass matrices  $M_u$  and  $M_d$  exceeds the number of observables. Furthermore we have to show that a Higgs potential exists allowing us to realize the desired VEV structure. In the simplest case we assume that all Higgs fields are  $SU(2)_L$  doublets as the Higgs field in the SM.

### 4.1 $D_7$ Assignments for Quarks

Here we present ways to produce the two matrix structures  $M_4$  and  $M_5$  shown in Eq.(4) and Eq.(5) with the help of the dihedral group  $D_7$ . Choosing  $D_7$  as flavor symmetry leaves us the possibility of either determining  $|V_{us}|$  or  $|V_{cd}|$  in terms of group theoretical quantities as  $\cos(\frac{3\pi}{7})$ .

#### 4.1.1 Matrix Structure $M_4$

For  $M_4$ , we assign the quarks to

$$Q_1 \sim \underline{\mathbf{1}}_1, \quad \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} \sim \underline{\mathbf{2}}_1, \quad u_1^c, d_1^c \sim \underline{\mathbf{1}}_2, \quad u_{2,3}^c, d_{2,3}^c \sim \underline{\mathbf{1}}_1 \quad (11)$$

under  $D_7$ . In this way we can generate the matrix structure found in Eq.(4) and Eq.(5) for the down as well as the up quarks. Thereby we assume that the theory contains Higgs doublet fields transforming as  $\underline{\mathbf{1}}_1$  and  $\underline{\mathbf{2}}_1$ , which we call  $H_s$  and  $H_{1,2}$ . As the relation between the mixing parameters of  $V_{CKM}$  and the group theoretical indices only arises, if the flavor symmetry  $D_7$  is broken down to a subgroup  $Z_2 = \langle BA^{m_u} \rangle$  by fields which couple to up quarks, while it is broken down to  $Z_2 = \langle BA^{m_d} \rangle$  with  $m_d \neq m_u$  by fields coupling to down quarks, we need an extra symmetry to perform this separation. In the SM this can be achieved by a  $Z_2^{(aux)}$  symmetry:

$$d_i^c \rightarrow -d_i^c \quad \text{and} \quad H_s^d \rightarrow -H_s^d, \quad H_i^d \rightarrow -H_i^d \quad (12)$$



while all other fields  $Q_i$ ,  $u_i^c$ ,  $H_s^u$  and  $H_{1,2}^u$  are invariant under  $Z_2^{(aux)}$ . Note that in principle also a Higgs field transforming as  $\underline{\mathbf{1}}_2$  under  $D_7$  could couple directly to the quarks. However, if this field acquires a non-vanishing VEV, its VEV breaks the residual  $Z_2$  generated by  $\langle \text{BA}^m \rangle$ . Therefore Higgs fields  $\sim \underline{\mathbf{1}}_2$  either do not get a VEV or do not exist in the model at all. In both cases they are not relevant in the discussion of the fermion mass matrices. So, we deal with six Higgs fields coupling to the fermions,  $H_s^u \sim (\underline{\mathbf{1}}_1, +1)$ ,  $H_{1,2}^u \sim (\underline{\mathbf{2}}_1, +1)$  and  $H_s^d \sim (\underline{\mathbf{1}}_1, -1)$ ,  $H_{1,2}^d \sim (\underline{\mathbf{2}}_1, -1)$  under  $D_7 \times Z_2^{(aux)}$ . The matrices are of the form:

$$M_u = \begin{pmatrix} 0 & y_1^u \langle H_s^u \rangle^* & y_2^u \langle H_s^u \rangle^* \\ y_3^u \langle H_1^u \rangle^* & y_4^u \langle H_1^u \rangle^* & y_5^u \langle H_1^u \rangle^* \\ -y_3^u \langle H_2^u \rangle^* & y_4^u \langle H_2^u \rangle^* & y_5^u \langle H_2^u \rangle^* \end{pmatrix} \quad \text{and} \quad M_d = \begin{pmatrix} 0 & y_1^d \langle H_s^d \rangle & y_2^d \langle H_s^d \rangle \\ y_3^d \langle H_2^d \rangle & y_4^d \langle H_2^d \rangle & y_5^d \langle H_2^d \rangle \\ -y_3^d \langle H_1^d \rangle & y_4^d \langle H_1^d \rangle & y_5^d \langle H_1^d \rangle \end{pmatrix}$$

where  $y_i^{u,d}$  denote Yukawa couplings. The VEV structure is taken to be:

$$\langle H_s^{d,u} \rangle > 0, \quad \langle H_1^d \rangle = \langle H_2^d \rangle = v_d, \quad \langle H_1^u \rangle = v_u e^{-\frac{3\pi i}{7}} \quad \text{and} \quad \langle H_2^u \rangle = v_u e^{\frac{3\pi i}{7}}$$

with  $v_d > 0$  and  $v_u > 0$ . The VEVs are required to be real apart from the phase  $\pm \frac{3\pi}{7}$  which is necessary for the correct breaking to the desired subgroup of  $D_7$ .

The parameters  $A, B, \dots$  shown in Eq.(4) and Eq.(5) can be written in terms of Yukawa couplings and VEVs:

$$A_u = y_1^u \langle H_s^u \rangle, \quad B_u = y_2^u \langle H_s^u \rangle, \quad C_u = y_3^u v_u e^{-\frac{3\pi i}{7}}, \quad D_u = y_4^u v_u e^{-\frac{3\pi i}{7}}, \quad E_u = y_5^u v_u e^{-\frac{3\pi i}{7}}, \\ A_d = y_1^d \langle H_s^d \rangle, \quad B_d = y_2^d \langle H_s^d \rangle, \quad C_d = y_3^d v_d, \quad D_d = y_4^d v_d, \quad E_d = y_5^d v_d$$

together with  $\phi_u = \frac{6\pi}{7}$  ( $m_u = 3$ ),  $\phi_d = 0$  ( $m_d = 0$ ) and  $j = 1$ , as the left-handed quark doublets of the second and third generation transform as  $\underline{\mathbf{2}}_1$ . The preserved  $Z_2$  subgroups are generated by  $\text{BA}^3$  and  $\text{B}$  in the up and the down quark sector, respectively. As we have not fixed the ordering of the mass eigenvalues, the question which of the elements of  $V_{CKM}$  is determined by group theoretical quantities to be  $\cos(\frac{3\pi}{7})$  cannot be answered at this point.

#### 4.1.2 Matrix Structure $M_5$

For the case of  $M_5$ , see Eq.(4) and Eq.(5), we can assign the quarks to:

$$Q_1, u_1^c, d_1^c \sim \underline{\mathbf{1}}_1, \quad \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix}, \begin{pmatrix} u_2^c \\ u_3^c \end{pmatrix}, \begin{pmatrix} d_2^c \\ d_3^c \end{pmatrix} \sim \underline{\mathbf{2}}_1 \quad (13)$$

under  $D_7$ . We then need five Higgs fields for each sector, i.e. for the up and the down quarks. These transform as

$$H_s^u \sim (\underline{\mathbf{1}}_1, +1), \quad \begin{pmatrix} H_1^u \\ H_2^u \end{pmatrix} \sim (\underline{\mathbf{2}}_1, +1), \quad \begin{pmatrix} h_1^u \\ h_2^u \end{pmatrix} \sim (\underline{\mathbf{2}}_2, +1) \\ H_s^d \sim (\underline{\mathbf{1}}_1, -1), \quad \begin{pmatrix} H_1^d \\ H_2^d \end{pmatrix} \sim (\underline{\mathbf{2}}_1, -1), \quad \begin{pmatrix} h_1^d \\ h_2^d \end{pmatrix} \sim (\underline{\mathbf{2}}_2, -1)$$

where we again assumed the existence of an extra  $Z_2^{(aux)}$  symmetry. Under this  $Z_2^{(aux)}$  the quarks transform in the same way as in the example above, i.e. only the down quarks  $d_i^c$  acquire a sign. The mass matrices are then in terms of Yukawa couplings and VEVs:

$$M_u = \begin{pmatrix} y_1^u \langle H_s^u \rangle^* & y_2^u \langle H_1^u \rangle^* & y_2^u \langle H_2^u \rangle^* \\ y_3^u \langle H_1^u \rangle^* & y_5^u \langle h_1^u \rangle^* & y_4^u \langle H_s^u \rangle^* \\ y_3^u \langle H_2^u \rangle^* & y_4^u \langle H_s^u \rangle^* & y_5^u \langle h_2^u \rangle^* \end{pmatrix} \quad \text{and} \quad M_d = \begin{pmatrix} y_1^d \langle H_s^d \rangle & y_2^d \langle H_2^d \rangle & y_2^d \langle H_1^d \rangle \\ y_3^d \langle H_2^d \rangle & y_5^d \langle h_2^d \rangle & y_4^d \langle H_s^d \rangle \\ y_3^d \langle H_1^d \rangle & y_4^d \langle H_s^d \rangle & y_5^d \langle h_1^d \rangle \end{pmatrix}$$

where  $y_i^{u,d}$  denote Yukawa couplings. The VEV structure is assumed to be:

$$\begin{aligned} \langle H_s^{d,u} \rangle &> 0, \quad \langle H_1^d \rangle = \langle H_2^d \rangle = v_d, \quad \langle h_1^d \rangle = \langle h_2^d \rangle = w_d, \\ \langle H_1^u \rangle &= v_u e^{-\frac{3\pi i}{7}}, \quad \langle H_2^u \rangle = v_u e^{\frac{3\pi i}{7}}, \quad \langle h_1^u \rangle = w_u e^{-\frac{6\pi i}{7}} \quad \text{and} \quad \langle h_2^u \rangle = w_u e^{\frac{6\pi i}{7}} \end{aligned}$$

with  $v_{d,u} > 0$  and  $w_{d,u} > 0$ . As above we only consider real values for the VEVs apart from the phases which are required by the desire to break down to a certain subgroup of  $D_7$ .

Compared to the form of  $M_5$  given in Eq.(4) and Eq.(5) we see that the parameters  $A, B, \dots$  are given by:

$$\begin{aligned} A_u &= y_1^u \langle H_s^u \rangle, \quad B_u = y_3^u v_u e^{-\frac{3\pi i}{7}}, \quad C_u = y_2^u v_u e^{-\frac{3\pi i}{7}}, \quad D_u = y_5^u w_u e^{-\frac{6\pi i}{7}}, \quad E_u = y_4^u \langle H_s^u \rangle, \\ A_d &= y_1^d \langle H_s^d \rangle, \quad B_d = y_3^d v_d, \quad C_d = y_2^d v_d, \quad D_d = y_5^d w_d, \quad E_d = y_4^d \langle H_s^d \rangle \end{aligned}$$

together with  $\phi_u = \frac{6\pi}{7}$  ( $m_u = 3$ ),  $\phi_d = 0$  ( $m_d = 0$ ) and  $j = k = 1$  for up as well as down quarks, since all generations transform as  $\underline{\mathbf{1}}_1 + \underline{\mathbf{2}}_1$  in this setup. Therefore the preserved subgroups in the up and down quark sector are again  $Z_2 = \langle B A^3 \rangle$  and  $Z_2 = \langle B \rangle$ .

Note that the shown assignments are not unique, since it is also possible to use another two-dimensional representation instead of  $\underline{\mathbf{2}}_1$  for the fermions. Obviously, then also the transformation properties of the Higgs fields have to be changed accordingly.

From the viewpoint of unification the second assignment in which the left-handed as well as the left-handed conjugate fields transform as  $\underline{\mathbf{1}} + \underline{\mathbf{2}}$  is more desirable. However in this case we need at least five Higgs fields transforming as  $\underline{\mathbf{1}}_1, \underline{\mathbf{2}}_i, \underline{\mathbf{2}}_j$  with  $i \neq j$  in order to arrive at the matrix structure  $M_5$ . As we have to separate the up quark from the down quark sector, i.e. have to have Higgs fields which either couple to up quarks or down quarks, we need at least ten such fields. Since we want to show the minimal model, we constrain ourselves to the first case, i.e. matrix structure  $M_4$ , in the following numerical study and the study of the corresponding Higgs potential and only give a numerical solution for the second matrix structure  $M_5$ .

## 4.2 Numerical Analysis of Quark Masses and Mixing Angles

### 4.2.1 Matrix Structure $M_4$

Coming to our numerical results we take all VEVs to have the same absolute value 61.5 GeV which equals the electroweak scale 174 GeV divided by  $\sqrt{8}$ , because our complete model includes eight Higgs fields<sup>4</sup>. The Yukawa couplings are taken to be

$$\begin{aligned} y_1^u &= 1.07967 \cdot e^{i(-2.17704)}, \quad y_2^u = 2.55955 \cdot e^{i(1.41609)}, \quad y_3^u = 1.9546 \cdot 10^{-5} \cdot e^{i(2.43366)}, \\ y_4^u &= 3.89557 \cdot 10^{-2} \cdot e^{i(-2.28452)}, \quad y_5^u = 7.47229 \cdot 10^{-2} \cdot e^{i(1.2469)}, \\ y_1^d &= 2.52251 \cdot 10^{-2} \cdot e^{i(3.00267)}, \quad y_2^d = 3.92611 \cdot 10^{-2} \cdot e^{i(-2.29202)}, \quad y_3^d = 6.20874 \cdot 10^{-4} \cdot e^{i(-0.54014)}, \\ y_4^d &= 8.95471 \cdot 10^{-5} \cdot e^{i(-2.13972)}, \quad y_5^d = 1.04917 \cdot 10^{-4} \cdot e^{i(-1.59912)} \end{aligned}$$

The values of the quark masses are then

$$\begin{aligned} m_u &= 0.0017 \text{ GeV}, \quad m_c = 0.62 \text{ GeV}, \quad m_t = 171 \text{ GeV}, \\ m_d &= 0.003 \text{ GeV}, \quad m_s = 0.054 \text{ GeV}, \quad m_b = 2.87 \text{ GeV} \end{aligned}$$

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<sup>4</sup>The additional two Higgs fields which do not couple to the fermions directly, are necessary in order to break accidental symmetries present in the Higgs potential which we discuss in Section 5. The equality of the VEVs is motivated by our numerical study of the Higgs potential which clearly prefers solutions in which the VEVs are of the same order, otherwise severe fine-tunings of the parameters in the potential are necessary. However, this does not exclude in general the possibility that for example  $m_b \ll m_t$  could be explained by a hierarchy among the VEVs of the Higgs fields coupling only to up quarks and those coupling only to down ones.

which correspond to the values given at  $M_Z$  [7]. For  $V_{CKM}$ , we find:

$$|V_{CKM}| = \begin{pmatrix} 0.97492 & 0.2225 & 3.95 \times 10^{-3} \\ 0.2224 & 0.97404 & 42.23 \times 10^{-3} \\ 8.11 \times 10^{-3} & 41.64 \times 10^{-3} & 0.9991 \end{pmatrix}$$

and  $J_{CP} = 3.09 \times 10^{-5}$ . All these values are within a 10 % error range [5]. Furthermore  $|V_{us}|$  is given by  $\cos(\frac{3\pi}{7}) = 0.2225$ . Due to the ordering of the eigenvalues the mass of the strange as well as the one of the up quark is determined by  $\sqrt{2}|C_d|$  and  $\sqrt{2}|C_u|$ , respectively. They therefore correspond to the eigenvalue  $(c - d)$  in the language of Section 2.

The Yukawa couplings lie in the range  $10^{-5} \dots 1$  due to the strong hierarchy of the quark masses. However this can be explained by the Froggatt-Nielsen mechanism [8]. The quarks transform in the following way:

$$q_{FN}(Q_1) = +1, \quad q_{FN}(Q_{2,3}) = +2, \quad q_{FN}(d_{1,2,3}^c) = 0 \quad q_{FN}(u_1^c) = +1, \quad q_{FN}(u_{2,3}^c) = -1$$

under the additional  $U(1)_{FN}$  symmetry. As usual we assume a gauge singlet  $\vartheta$  with  $q_{FN}(\vartheta) = -1$  which neither transforms under  $D_7$  nor under  $Z_2^{(aux)}$  and which acquires a VEV  $\langle \vartheta \rangle$  at a large energy scale <sup>5</sup>. According to the choice of the  $U(1)_{FN}$  charges only the Yukawa couplings  $y_{1,2}^u$  exist at tree level while all other couplings require some insertion of the  $\vartheta$  field, i.e. become non-renormalizable involving some power of  $\frac{\langle \vartheta \rangle}{\Lambda}$  with  $\Lambda$  being the cutoff scale of the theory. One can then re-write the given Yukawa couplings  $y_i^{u,d}$  in terms of new couplings  $\tilde{y}_i^{u,d}$  and  $\epsilon \equiv \frac{\langle \vartheta \rangle}{\Lambda} = 3 \cdot 10^{-2}$  as  $y_i^{u,d} = \tilde{y}_i^{u,d} \epsilon^x$  where  $x$  is determined by the charges of the quark fields under the  $U(1)_{FN}$ . The values of  $\tilde{y}_i^{u,d}$  are then all of natural size:

$$\begin{aligned} \tilde{y}_1^u &= 1.07967 \cdot e^{i(-2.17704)}, \quad \tilde{y}_2^u = 2.55955 \cdot e^{i(1.41609)}, \quad \tilde{y}_3^u = 0.723926 \cdot e^{i(2.43366)}, \\ \tilde{y}_4^u &= 1.29852 \cdot e^{i(-2.28452)}, \quad \tilde{y}_5^u = 2.49076 \cdot e^{i(1.2469)}, \\ \tilde{y}_1^d &= 0.840837 \cdot e^{i(3.00267)}, \quad \tilde{y}_2^d = 1.3087 \cdot e^{i(-2.29202)}, \quad \tilde{y}_3^d = 0.68986 \cdot e^{i(-0.54014)}, \\ \tilde{y}_4^d &= 0.099497 \cdot e^{i(-2.13972)}, \quad \tilde{y}_5^d = 0.116574 \cdot e^{i(-1.59912)} \end{aligned}$$

#### 4.2.2 Matrix Structure $M_5$

For the second matrix structure  $M_5$ , we also performed a numerical study with the mass matrix structure given above and found for example the following possible values for the parameters  $A_{u,d}, B_{u,d}, \dots$ :

$$\begin{aligned} A_u &= 40.40221 \cdot e^{i(0.185452)}, \quad B_u = 0.238084 \cdot e^{i(-2.99845)}, \quad C_u = 117.4875 \cdot e^{i(-0.234118)}, \\ D_u &= 0.420584 \cdot e^{i(-3.13931)}, \quad E_u = 0.984542 \cdot e^{i(-0.849532)}, \\ A_d &= 2.233447 \cdot e^{i(-1.91017)}, \quad B_d = 0.051223 \cdot e^{i(-3.05165)}, \quad C_d = 1.271448 \cdot e^{i(-0.751605)}, \\ D_d &= 0.058343 \cdot e^{i(-2.41411)}, \quad E_d = 0.056221 \cdot e^{i(-2.37708)}. \end{aligned}$$

All values are given in GeV. The phases  $\phi_{u,d}$  can be chosen to be  $\phi_u = \frac{6\pi}{7}$  and  $\phi_d = 0$ . The quark masses are then:

$$\begin{aligned} m_u &= 0.0017 \text{ GeV}, \quad m_c = 0.62 \text{ GeV}, \quad m_t = 171 \text{ GeV}, \\ m_d &= 0.003 \text{ GeV}, \quad m_s = 0.054 \text{ GeV}, \quad m_b = 2.87 \text{ GeV} \end{aligned}$$

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<sup>5</sup>Here we assume that the  $U(1)_{FN}$  is broken explicitly in other parts of the Lagrangian so that no Goldstone boson arises when  $\vartheta$  gets a VEV.

and the absolute values of  $V_{CKM}$ :

$$|V_{CKM}| = \begin{pmatrix} 0.97489 & 0.2226 & 3.95 \times 10^{-3} \\ 0.2225 & 0.97401 & 42.23 \times 10^{-3} \\ 8.11 \times 10^{-3} & 41.64 \times 10^{-3} & 0.9991 \end{pmatrix}$$

together with  $J_{CP} = 3.09 \times 10^{-5}$ . These values match the experimental results quite well. Note here that this time not  $|V_{us}|$ , but now  $|V_{cd}|$  is given in terms of the group theoretical indices, i.e.  $|V_{cd}| = \cos(\frac{3\pi}{7}) = 0.2225$ . Since  $|V_{us}|_{exp} \approx |V_{cd}|_{exp}$ ,  $|V_{cd}|$  is now 2% below its experimental value. This is due to the fact that the eigenvalue  $(c - d)$  introduced in Section 2 is given by  $m_c$  in the up quark and by  $m_d$  in the down quark sector. These masses can be expressed in a simple way in terms of the parameters  $D_{u,d}$  and  $E_{u,d}$ , namely  $m_c = |D_u - E_u e^{-i\phi_u k}|$  and  $m_d = |D_d - E_d e^{i\phi_d k}|$  with  $\phi_u = \frac{6\pi}{7}$ ,  $\phi_d = 0$  and  $k = 1$ <sup>6</sup> as shown above. Also here the hierarchy among the parameters  $A_{u,d}, B_{u,d}, \dots$  which are products of Yukawa couplings and VEVs may not be explained by a hierarchy among the VEVs. Assuming that all VEVs have the same absolute value, i.e.  $\frac{174}{\sqrt{10}} \text{ GeV} \approx 55 \text{ GeV}$ <sup>7</sup>, an additional  $U(1)_{FN}$  is responsible for the fermion mass hierarchy. The quarks have the  $U(1)_{FN}$  charges:

$$q_{FN}(Q_{2,3}) = +2 \quad \text{and} \quad q_{FN}(d_1^c) = +1$$

and the other fields have zero charge. The parameter  $\epsilon = \frac{\langle \vartheta \rangle}{\Lambda}$  ( $\vartheta$ :  $U(1)_{FN}$  breaking field with  $q_{FN}(\vartheta) = -1$ ,  $\Lambda$ : cutoff scale) should be around  $8 \cdot 10^{-2}$ .

## 5 Higgs Sector

In this section, the Higgs sector belonging to the first numerical example given in Section 4.1.1 is discussed. As already mentioned above, we concentrate on a multi-Higgs doublet potential. We are aware of the fact that such multi-Higgs doublet models usually suffer from the problem that large FCNCs are induced by the additional Higgs fields. However, as a proof of principle that we can produce our required VEV configuration the consideration of such a multi-Higgs doublet model seems to be reasonable. The minimal number of fields needed in order to produce the fermion mass matrices is  $2 \times 3$ , i.e. the model includes three Higgs  $SU(2)_L$  doublets, called  $H_s^d$  and  $H_{1,2}^d$ , coupling to the down quarks and three doublets,  $H_s^u$  and  $H_{1,2}^u$ , coupling to the up quarks. This separation is necessary, since the key point of the study lies in the fact that a sizeable mixing angle, like the Cabibbo angle, can only arise from preserved (non-trivial) subgroups, if we break to different (directions of) subgroups in the down and up quark sector. The subgroups correspond to different VEV configurations of the Higgs doublets  $\{H_s^d, H_{1,2}^d\}$  and  $\{H_s^u, H_{1,2}^u\}$ . An additional  $Z_2^{(aux)}$  symmetry is introduced in order to perform the separation. According to Eq.(12) in Section 4.1.1 the Higgs fields coupling to the down quarks acquire a sign under  $Z_2^{(aux)}$ . We first construct the three Higgs doublet potential with Higgs fields  $H_s \sim \underline{1}_1$  and  $\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \sim \underline{2}_1$ .

<sup>6</sup>k is here the same for the up and down quark sector due to the choice of the transformation properties of the left-handed conjugate fields  $u_{2,3}^c$  and  $d_{2,3}^c$ . Note, however, that they could be in principle different.

<sup>7</sup>Here we assume the existence of only the ten Higgs fields which couple to the fermions in order to produce the mass matrix structure  $M_5$ , as it seems unlikely that there are accidental symmetries in the Higgs potential which would enforce the existence of further Higgs fields.

The potential has the form: <sup>8</sup>

$$\begin{aligned}
V_3(H_s, H_i) = & -\mu_s^2 H_s^\dagger H_s - \mu_D^2 \sum_{i=1}^2 H_i^\dagger H_i + \lambda_s (H_s^\dagger H_s)^2 + \lambda_1 \left( \sum_{i=1}^2 H_i^\dagger H_i \right)^2 \\
& + \lambda_2 \left( H_1^\dagger H_1 - H_2^\dagger H_2 \right)^2 + \lambda_3 |H_1^\dagger H_2|^2 \\
& + \sigma_1 (H_s^\dagger H_s) \left( \sum_{i=1}^2 H_i^\dagger H_i \right) + \{ \sigma_2 (H_s^\dagger H_1) (H_s^\dagger H_2) + \text{h.c.} \} + \sigma_3 \sum_{i=1}^2 |H_s^\dagger H_i|^2
\end{aligned} \tag{14}$$

As already shown in [9] and also mentioned in [10], this potential has an additional  $U(1)$  symmetry, i.e. there exists a further  $U(1)$  symmetry in the potential apart from the  $U(1)_Y$  symmetry. This further symmetry is necessarily broken by our desired VEV structure such that a massless Goldstone boson appears in the Higgs spectrum which is not eaten by a gauge boson. This problem cannot be solved by taking into account the whole potential for all six Higgs fields, since even if the terms, coupling the fields  $H_s^u$ ,  $H_{1,2}^u$  and  $H_s^d$ ,  $H_{1,2}^d$  together, are included, we find an accidental  $U(1)$  symmetry in the potential. Therefore we have to enlarge the Higgs sector by further Higgs fields in order to create new  $D_7$  invariant couplings which break this accidental symmetry explicitly. We find that this can be done in the simplest way by adding two Higgs fields transforming as  $\underline{\mathbf{2}}_2$  under  $D_7$ . Due to their transformation properties they do not directly couple to the fermions (see Section 4.1.1). We decided to add two such fields to the three Higgs fields which couple to the down quarks. Therefore the model contains eight Higgs doublet fields in total: three of them couple to up and three of them to down quarks, while the other two ones are needed for a viable Higgs sector:

$$\begin{aligned}
H_s^u & \sim (\underline{\mathbf{1}}_1, +1), \quad \begin{pmatrix} H_1^u \\ H_2^u \end{pmatrix} \sim (\underline{\mathbf{2}}_1, +1), \\
H_s^d & \sim (\underline{\mathbf{1}}_1, -1), \quad \begin{pmatrix} H_1^d \\ H_2^d \end{pmatrix} \sim (\underline{\mathbf{2}}_1, -1) \quad \text{and} \quad \begin{pmatrix} \chi_1^d \\ \chi_2^d \end{pmatrix} \sim (\underline{\mathbf{2}}_2, -1).
\end{aligned} \tag{15}$$

under  $D_7 \times Z_2^{(aux)}$ . The complete potential consists of three parts:

$$V = V_u + V_d + V_{mixed} \tag{16}$$

where  $V_u$  denotes the part of the potential which only contains Higgs fields coupling to the up quarks,  $V_d$  contains the five Higgs fields which have a non-vanishing  $Z_2^{(aux)}$  charge (three of them give masses to the down quarks), while  $V_{mixed}$  consists of all other terms. The explicit form of the potential is given in Appendix C.

The VEV structure of the fields  $H_s^{d,u}$  and  $H_{1,2}^{d,u}$  is determined by our desire to break down to two distinct  $Z_2$  subgroups in the up and the down quark sector (see Section 4.1.1):

$$\langle H_s^{d,u} \rangle > 0, \quad \langle H_1^d \rangle = \langle H_2^d \rangle = v_d, \quad \langle H_1^u \rangle = v_u e^{-\frac{3\pi i}{7}} \quad \text{and} \quad \langle H_2^u \rangle = v_u e^{\frac{3\pi i}{7}}$$

with  $v_d > 0$  and  $v_u > 0$ . In contrast to this, the VEV structure of the fields  $\chi_{1,2}^d$  is not fixed in this way. However, in order to preserve the  $Z_2$  subgroup generated by B not only through the VEVs of the fields  $H_s^d$  and  $H_{1,2}^d$ , but also by the VEVs of the fields  $\chi_{1,2}^d$ ,  $\langle \chi_1^d \rangle = \langle \chi_2^d \rangle > 0$  will be assumed in the following (see Section 2).

We proceed in the following way in order to find a minimum of this potential which allows for

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<sup>8</sup>Note that  $\sigma_2$  is complex, but it can be made real by appropriate redefinition of the field  $H_s$ , for example.

our choice of VEVs: first we treat  $V_u$  and  $V_d$  separately to find a viable solution for these two parts of the potential. Note that we can allow all parameters in the potential  $V_d$  to be real, as the VEVs of the corresponding Higgs fields are also real. Since  $V_u$  suffers from the above mentioned accidental  $U(1)$  symmetry, we find a fourth massless particle in the Higgs mass spectrum. In a second step we add as many terms as necessary from  $V_{mixed}$  to get a minimum of the whole potential  $V$  which does not have more than the usual three Goldstone bosons. It turns out that it is sufficient to take into account three terms in addition to  $V_u$  and  $V_d$  to get a viable solution. The terms are of the form:

$$\kappa_2 \left( H_s^{u\dagger} H_s^d \right)^2 + \kappa_5 \left( \sum_{i=1}^2 H_i^{u\dagger} H_i^d \right)^2 + \kappa_{19} \left( H_s^{u\dagger} H_s^d \right) \left( \sum_{i=1}^2 H_i^{u\dagger} H_i^d \right) + \text{h.c.} \subset V_{mixed}$$

Note that we take all VEVs to have the same absolute value, since this considerably simplifies the search for a numerical solution, as a fine-tuning of the parameters in the Higgs potential is avoided. However, in principle other solutions should also be possible, e.g. the fact that the up quarks are much heavier than the down ones could be explained by assuming that the VEVs of the fields  $H_s^u$ ,  $H_{1,2}^u$  are (much) larger than the ones of the fields  $H_s^d$ ,  $H_{1,2}^d$ .

Finally, let us mention that the resulting Higgs masses are usually in between 50 and 500 GeV. These values are either not favored by the constraints coming from FCNCs or already excluded by direct searches. There are two reasons for the too low Higgs masses: on the one hand  $V_u$  contains an accidental symmetry and on the other hand all mass parameters of the potential are chosen to be of natural order, i.e. to be around the electroweak scale. Additionally, all quartic couplings of the potential must be perturbative. However, as already mentioned above, this model is not intended to be fully realistic. Adding  $D_7$  breaking soft masses to the potential might allow to push the masses of the additional Higgs particles above 10 TeV.

The rest of the discussion of the potential is delegated to Appendix C where we present a numerical solution for the parameters of the Higgs potential and the resulting Higgs masses.

## 6 Ways to generate $\theta_C$ only

In the preceding sections we confined ourselves to cases in which all mixing angles can be reproduced at tree level. Therefore we only discussed the matrix structures  $M_4$  and  $M_5$  of Eq.(4) and Eq.(5). However,  $\theta_{13}^q$  and  $\theta_{23}^q$  are roughly an order of magnitude smaller than the Cabibbo angle  $\theta_C \equiv \theta_{12}^q$  which gives reason for also considering matrix structures which lead to only  $\theta_C \neq 0$  at LO. For this a block matrix structure (with correlated elements), which we introduced in Eq.(3), is suitable. Such a structure can be achieved in at least two different ways: *a.*) we can simply omit some of the Higgs fields which are in principle allowed a VEV in order to arrive at the zero elements of the mass matrix; *b.*) we can demand that the preserved subgroup is not just a  $Z_2$  symmetry, but a dihedral group  $D_q$  with  $q > 1$ <sup>9</sup>. Note that due to the choice of the scalar fields in case *a.*) the results of such a model are a bit arbitrary, since the structure of the mass matrices and therefore the mixing pattern is not fully determined by the fermions and the flavor symmetry alone. In the following, we show examples for the two cases. For case *a.*) the simplest example is probably the one in which the quarks transform as

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \begin{pmatrix} u_1^c \\ u_2^c \end{pmatrix}, \begin{pmatrix} d_1^c \\ d_2^c \end{pmatrix} \sim \underline{\mathbf{2}}_1, \quad Q_3, u_3^c, d_3^c \sim \underline{\mathbf{1}}_1$$

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<sup>9</sup>The general results of the mass matrices from the various preserved subgroups can be found in [1].

under  $D_7$  and we assume that there exist two sets of three Higgs fields transforming as

$$H_s^u \sim (\underline{\mathbf{1}}_1, +1), \quad \begin{pmatrix} h_1^u \\ h_2^u \end{pmatrix} \sim (\underline{\mathbf{2}}_2, +1), \quad H_s^d \sim (\underline{\mathbf{1}}_1, -1), \quad \begin{pmatrix} h_1^d \\ h_2^d \end{pmatrix} \sim (\underline{\mathbf{2}}_2, -1)$$

under  $D_7 \times Z_2^{(aux)}$ , one of them coupling to up and the other one coupling to down quarks. The additional  $Z_2^{(aux)}$  symmetry is the same as used above (see Section 4.1.1). Then the mass matrices are of the form:

$$M_u = \begin{pmatrix} y_3^u \langle h_1^u \rangle^* & y_2^u \langle H_s^u \rangle^* & 0 \\ y_2^u \langle H_s^u \rangle^* & y_3^u \langle h_2^u \rangle^* & 0 \\ 0 & 0 & y_1^u \langle H_s^u \rangle^* \end{pmatrix} \quad \text{and} \quad M_d = \begin{pmatrix} y_3^d \langle h_2^d \rangle & y_2^d \langle H_s^d \rangle & 0 \\ y_2^d \langle H_s^d \rangle & y_3^d \langle h_1^d \rangle & 0 \\ 0 & 0 & y_1^d \langle H_s^d \rangle \end{pmatrix}$$

Assuming the VEV structure:

$$\langle H_s^{d,u} \rangle > 0, \quad \langle h_1^d \rangle = \langle h_2^d \rangle = w_d, \quad \langle h_1^u \rangle = w_u e^{-\frac{6\pi i}{7}} \quad \text{and} \quad \langle h_2^u \rangle = w_u e^{\frac{6\pi i}{7}}$$

with  $w_{d,u} > 0$  we arrive at

$$M_u = \begin{pmatrix} y_3^u w_u e^{\frac{6\pi i}{7}} & y_2^u \langle H_s^u \rangle & 0 \\ y_2^u \langle H_s^u \rangle & y_3^u w_u e^{-\frac{6\pi i}{7}} & 0 \\ 0 & 0 & y_1^u \langle H_s^u \rangle \end{pmatrix} \quad \text{and} \quad M_d = \begin{pmatrix} y_3^d w_d & y_2^d \langle H_s^d \rangle & 0 \\ y_2^d \langle H_s^d \rangle & y_3^d w_d & 0 \\ 0 & 0 & y_1^d \langle H_s^d \rangle \end{pmatrix}$$

and

$$|V_{CKM}| = \begin{pmatrix} |\cos(\frac{\pi}{14})| & |\cos(\frac{3\pi}{7})| & 0 \\ |\cos(\frac{3\pi}{7})| & |\cos(\frac{\pi}{14})| & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 0.97493 & 0.2225 & 0 \\ 0.2225 & 0.97493 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (17)$$

The preserved subgroup is a  $Z_2$  in each sector which is generated by  $BA^3$  and  $B$  in the up quark and the down quark sector, respectively. The masses of the quarks are  $(m_u^2, m_c^2, m_t^2) = (|y_2^u \langle H_s^u \rangle + y_3^u w_u|^2, |y_2^u \langle H_s^u \rangle - y_3^u w_u|^2, |y_1^u \langle H_s^u \rangle|^2)$  and  $(m_d^2, m_s^2, m_b^2) = (|y_2^d \langle H_s^d \rangle - y_3^d w_d|^2, |y_2^d \langle H_s^d \rangle + y_3^d w_d|^2, |y_1^d \langle H_s^d \rangle|^2)$ , i.e. the mass of the third generation is solely determined by  $y_1^{u,d} \langle H_s^{u,d} \rangle$ . Note that if the VEVs of  $H_s^{u,d}$  are taken to be large in order to explain the large mass of the third generation, the Yukawa couplings  $y_2^{u,d}$  have to be suppressed. This might be viewed as fine-tuning. A possible solution is the assumption of an additional  $U(1)_{FN}$  as already used above or to consider the case *b*.) instead. The possibility to choose the two-dimensional representations for the left-handed and left-handed conjugate fields to be distinct from each other, such that the trivial representation  $\underline{\mathbf{1}}_1$  cannot be coupled to the first and second generation, does not exist, since in this case also two of the four zeros disappear. The reason can be found by looking at the Kronecker products shown in Appendix B. Actually, this setup is very similar to the one shown in Section 4.1.2. The main difference is the fact that now there are no Higgs fields transforming as  $\underline{\mathbf{2}}_1$  under  $D_7$ . The other difference is that the first and second generation of the left-handed and left-handed conjugate fields are unified into the doublet under the flavor group instead of the second and third one as done above. However, this only leads to a change in the appearance of the mass matrix itself, but does not have any phenomenological consequences, since this permutation of fields is cancelled in the mixing matrix. The existence of six instead of ten Higgs fields coupling to the fermions may be advantageous with regard to the problem of FCNCs mediated by these fields. The corresponding Higgs potential ought to be of the same form as the one discussed in Section 5.

The second case *b.*) cannot be maintained with the flavor group  $D_7$  which we used throughout this work, since it only contains  $Z_q$  groups as subgroups, but no dihedral ones  $D_q$ ,  $q > 1$ . Therefore we have to consider the group  $D_{14}$  instead. In the study of the  $V_{CKM}$  elements in Section 2 and Section 3  $D_{14}$  turned out to be the smallest group which is appropriate to describe the elements  $|V_{ud}|$  and  $|V_{cs}|$  in terms of group theoretical indices. As argued in Section 2 and Section 3 it can also be used in order to reproduce the  $D_7$  results, i.e. either  $|V_{us}| = |\cos(\frac{3\pi}{7})|$  or  $|V_{cd}| = |\cos(\frac{3\pi}{7})|$ . Here we just show a possible example in which  $D_{14}$  is broken to its subgroup  $D_2 = \langle A^7, BA^m \rangle$  ( $m = 0, 1, \dots, 6$ ) in order to reproduce a matrix of block structure. We assign the quarks to

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \begin{pmatrix} u_1^c \\ u_2^c \end{pmatrix}, \begin{pmatrix} d_1^c \\ d_2^c \end{pmatrix} \sim \underline{\mathbf{2}}_1, \quad Q_3, u_3^c, d_3^c \sim \underline{\mathbf{1}}_1$$

under  $D_{14}$ . According to the Kronecker products

$$\underline{\mathbf{1}}_1 \times \underline{\mathbf{2}}_1 = \underline{\mathbf{2}}_1 \quad \text{and} \quad \underline{\mathbf{2}}_1 \times \underline{\mathbf{2}}_1 = \underline{\mathbf{1}}_1 + \underline{\mathbf{2}}_2 + \underline{\mathbf{2}}_2$$

the Higgs fields which can in principle couple to form  $D_{14}$ -invariants have to transform as  $\underline{\mathbf{1}}_1$ ,  $\underline{\mathbf{1}}_2$ ,  $\underline{\mathbf{2}}_1$  and  $\underline{\mathbf{2}}_2$ . However,  $\underline{\mathbf{1}}_2$  is not allowed a VEV and the representation index  $j$  of  $\underline{\mathbf{2}}_j$  has to be even <sup>10</sup>. Therefore we take

$$H_s^u \sim \underline{\mathbf{1}}_1, \quad \begin{pmatrix} H_1^u \\ H_2^u \end{pmatrix} \sim \underline{\mathbf{2}}_2, \quad H_s^d \sim \underline{\mathbf{1}}_1 \quad \text{and} \quad \begin{pmatrix} H_1^d \\ H_2^d \end{pmatrix} \sim \underline{\mathbf{2}}_2$$

(with implicit  $Z_2^{(aux)}$  assignment as above) and arrive at the matrix forms which are exactly the same as given above for case *a.*) <sup>11</sup>, if we assume the VEVs to be

$$\langle H_s^{u,d} \rangle > 0, \quad \langle H_1^u \rangle = v_u e^{-\frac{6\pi i}{7}}, \quad \langle H_2^u \rangle = v_u e^{\frac{6\pi i}{7}}, \quad \langle H_1^d \rangle = \langle H_2^d \rangle = v_d$$

The subgroups which are preserved by the VEVs in the up and down quark sector are then of the form  $D_2 = \langle A^7, BA^m \rangle$  with  $m_u = 6$  for the up quarks and  $m_d = 0$  for down quarks. Also here the Higgs fields  $H_s^{d,u}$  couple to all three generations. In order to avoid this one can assign the quarks to different  $D_{14}$  representations, e.g.

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \sim \underline{\mathbf{2}}_1, \quad \begin{pmatrix} u_1^c \\ u_2^c \end{pmatrix}, \begin{pmatrix} d_1^c \\ d_2^c \end{pmatrix} \sim \underline{\mathbf{2}}_3, \quad Q_3, u_3^c, d_3^c \sim \underline{\mathbf{1}}_1$$

Since  $\underline{\mathbf{2}}_1 \times \underline{\mathbf{2}}_3$  decomposes into  $\underline{\mathbf{2}}_2$  and  $\underline{\mathbf{2}}_4$  in  $D_{14}$ , the  $1-2$  sub-block of the mass matrices is produced by the VEVs of Higgs fields belonging to  $D_{14}$  doublets instead of the singlet  $H_s^{u,d}$ . As the indices of the representations  $\underline{\mathbf{2}}_2$  and  $\underline{\mathbf{2}}_4$  are even, they are allowed a VEV by the requirement to preserve a  $D_2$  subgroup of  $D_{14}$ . We need five Higgs fields transforming as  $\underline{\mathbf{1}}_1 + \underline{\mathbf{2}}_2 + \underline{\mathbf{2}}_4$  for the down as well as the up quarks. The general form of the mass matrices reads

$$M_u = \begin{pmatrix} y_3^u \langle h_1^u \rangle^* & y_2^u \langle H_2^u \rangle^* & 0 \\ y_2^u \langle H_1^u \rangle^* & y_3^u \langle h_2^u \rangle^* & 0 \\ 0 & 0 & y_1^u \langle H_s^u \rangle^* \end{pmatrix} \quad \text{and} \quad M_d = \begin{pmatrix} y_3^d \langle h_2^d \rangle & y_2^d \langle H_1^d \rangle & 0 \\ y_2^d \langle H_2^d \rangle & y_3^d \langle h_1^d \rangle & 0 \\ 0 & 0 & y_1^d \langle H_s^d \rangle \end{pmatrix}$$

With the VEVs

$$\langle H_s^{d,u} \rangle > 0, \quad \langle H_1^d \rangle = \langle H_2^d \rangle = v_d, \quad \langle h_1^d \rangle = \langle h_2^d \rangle = w_d, \\ \langle H_1^u \rangle = v_u e^{-\frac{6\pi i}{7}}, \quad \langle H_2^u \rangle = v_u e^{\frac{6\pi i}{7}}, \quad \langle h_1^u \rangle = w_u e^{-\frac{12\pi i}{7}} \quad \text{and} \quad \langle h_2^u \rangle = w_u e^{\frac{12\pi i}{7}}$$

<sup>10</sup>In general this index must be divisible by the group index of the dihedral subgroup which should be preserved.

<sup>11</sup>The Clebsch Gordan coefficients necessary for the calculation of the mass matrices in  $D_{14}$  can be found in a general form in [1]. However, in this special case they coincide with those given for the group  $D_7$ .



for  $H_s^{u,d} \sim \mathbf{1_1}$ ,  $H_{1,2}^{u,d} \sim \mathbf{2_2}$  and  $h_{1,2}^{u,d} \sim \mathbf{2_4}$ , we can achieve

$$M_u = \begin{pmatrix} y_3^u w_u e^{\frac{12\pi i}{7}} & y_2^u v_u e^{\frac{-6\pi i}{7}} & 0 \\ y_2^u v_u e^{\frac{6\pi i}{7}} & y_3^u w_u e^{\frac{-12\pi i}{7}} & 0 \\ 0 & 0 & y_1^u \langle H_s^u \rangle \end{pmatrix} \quad \text{and} \quad M_d = \begin{pmatrix} y_3^d w_d & y_2^d v_d & 0 \\ y_2^d v_d & y_3^d w_d & 0 \\ 0 & 0 & y_1^d \langle H_s^d \rangle \end{pmatrix}$$

For  $(m_u^2, m_c^2, m_t^2) = (|y_2^u v_u + y_3^u w_u|^2, |y_2^u v_u - y_3^u w_u|^2, |y_1^u \langle H_s^u \rangle|^2)$  and  $(m_d^2, m_s^2, m_b^2) = (|y_2^d v_d - y_3^d w_d|^2, |y_2^d v_d + y_3^d w_d|^2, |y_1^d \langle H_s^d \rangle|^2)$  the CKM matrix is of the form as given in Eq.(17). Although the Higgs fields  $H_s^{u,d}$  couple in this setup only to the third generation and therefore can have a large VEV without spoiling the masses of the lighter quarks, there still exists a source of fine-tuning, since the uncorrelated parameters  $y_{2,3}^{d,u}$ ,  $v_{d,u}$  and  $w_{d,u}$  have to be arranged such that  $|y_2^d v_d - y_3^d w_d| \ll |y_2^d v_d + y_3^d w_d|$  for  $m_d \ll m_s$  and  $|y_2^u v_u + y_3^u w_u| \ll |y_2^u v_u - y_3^u w_u|$  for  $m_u \ll m_c$ . The preserved subgroups in the up and the down quark sector are again  $D_2 = \langle A^7, B A^6 \rangle$  and  $D_2 = \langle A^7, B \rangle$ , respectively.

## 7 Numerical Analysis of $V_{MNS}$

A similar analysis as done in the case of  $V_{CKM}$  can also be carried out for the lepton mixing matrix  $V_{MNS}$ . We assume that the neutrinos are Dirac particles as all the other fermions and that they have the same ordering as the other fermions, i.e. the neutrino mass spectrum is normally ordered. This allows us to use the matrix structures found in Appendix A also for  $V_{MNS}$ . Since the entries of  $V_{MNS}$  are not strongly restricted by experiments [11] (at  $3\sigma$ ):

$$|V_{MNS}^{(\text{range})}| = \begin{pmatrix} 0.79 - 0.88 & 0.47 - 0.61 & < 0.20 \\ 0.19 - 0.52 & 0.42 - 0.73 & 0.58 - 0.82 \\ 0.20 - 0.53 & 0.44 - 0.74 & 0.56 - 0.81 \end{pmatrix} \quad (18)$$

there are several more possibilities to accommodate the various matrix elements regarding the choice of the group index  $n$ , and the values  $m_l$ ,  $m_\nu$  and  $j$ . However, as we intend to build a model which includes quarks as well as leptons, we stick to the selected values of  $n$ ,  $n = 7$ ,  $n = 14$ , which fit the CKM matrix elements of the  $1 - 2$  sub-block best, if we restrict ourselves to small  $n$ . We check element by element of  $V_{MNS}$  whether we can put it into the form  $|\cos(\frac{l\pi}{7})|$  where  $l = 0, 1, 2, \dots, 6$  or  $|\cos(\frac{l\pi}{14})|$  with  $l = 0, 1, 2, \dots, 13$ . According to Eq.(18) all elements of the second and third row can be approximated by a cosine of the form  $|\cos(\frac{l\pi}{7})|$  ( $l = 0, 1, 2, \dots, 6$ ) or  $|\cos(\frac{l\pi}{14})|$  ( $l = 0, 1, 2, \dots, 13$ )<sup>12</sup>. We take into account all possibilities shown in Table 2 and perform a numerical fit of the mixing angles  $\theta_{12}$ ,  $\theta_{13}$  and  $\theta_{23}$ . In the fit procedure we compute the sines of the three mixing angles and compare these to the best fit values, which are  $\sin^2(\theta_{23}^{bf}) = 0.5$ ,  $\sin^2(\theta_{12}^{bf}) = 0.3$  and  $\sin^2(\theta_{13}^{bf}) = 0$  [12]<sup>13</sup>. Again, we replace the experimentally allowed  $2\sigma$  or  $3\sigma$  ranges by 10% ranges (around the best fit value). For  $\sin^2(\theta_{13})$  we consider two possible upper bounds:  $\sin^2(\theta_{13}) \leq 0.025$  which corresponds to the  $2\sigma$  bound [12] and a much more loose bound  $\sin^2(\theta_{13}) \leq 0.1$  being even larger than the  $4\sigma$  bound [12]. This is done, since the numerical study showed that loosening the bound on  $\sin^2(\theta_{13})$  leads to several more solutions. Our results for  $\sin^2(\theta_{13}) \leq 0.1$  are summarized in Table 3 where we also display the numerical values for  $\theta_l$ ,  $\theta_\nu$  and  $\alpha = \beta_l - \beta_\nu$  together with the resulting mixing angles and the (Dirac) CP phase  $\delta$ .

<sup>12</sup>We omit the trivial possibility that the (13) element can be approximated by 0.

<sup>13</sup>Note that these best fit values are not presented in the same global analysis as the above mentioned allowed  $3\sigma$  ranges for the elements of  $V_{MNS}$ . Nevertheless the deviations are very small such that we do not consider this to lead to a major difference in our numerical analysis.

Element ( $ij$ )	Possible cosines
(21)	$\cos(\frac{3\pi}{7}) (\approx 0.2225)$ , $\cos(\frac{5\pi}{14}) (\approx 0.4339)$
(22)	$\cos(\frac{5\pi}{14}) (\approx 0.4339)$ , $\cos(\frac{2\pi}{7}) (\approx 0.6235)$
(23)	$\cos(\frac{2\pi}{7}) (\approx 0.6235)$ , $\cos(\frac{3\pi}{14}) (\approx 0.7818)$
(31)	$\cos(\frac{3\pi}{7}) (\approx 0.2225)$ , $\cos(\frac{5\pi}{14}) (\approx 0.4339)$
(32)	$\cos(\frac{2\pi}{7}) (\approx 0.6235)$
(33)	$\cos(\frac{2\pi}{7}) (\approx 0.6235)$ , $\cos(\frac{3\pi}{14}) (\approx 0.7818)$

Table 2: Possibilities for the group theoretically determined element in  $V_{MNS}$ . Note that, e.g.  $\cos(\frac{3\pi}{7})$  equals  $\cos(\frac{6\pi}{14})$ , i.e. it could also be reproduced in the group  $D_{14}$  with  $j = 1$  and  $m_l - m_\nu = 6$  and not only in  $D_7$  with  $j = 1$  and  $m_l - m_\nu = 3$ . Furthermore, for example,  $\cos(\frac{4\pi}{7})$  is also included implicitly in the list, as  $|\cos(\frac{4\pi}{7})| = |\cos(\frac{3\pi}{7})|$ .

One can observe the following: there are some cosines listed in Table 2 for which no fit with  $\chi^2 < 1$  has been found. In all these cases the value of the fixed  $V_{MNS}$  element lies almost outside the ranges shown in Eq.(18), e.g. for the (23) element the possible cosines are  $\cos(\frac{2\pi}{7}) \approx 0.6235$  and  $\cos(\frac{3\pi}{14}) \approx 0.7818$  with the first being quite close to the lower bound (0.58) and the second one close to the upper one (0.82) of the allowed range. Furthermore, by having a closer look at the form of  $|V_{MNS}^{23}|$  given in Appendix A, one realizes that  $\tan(\theta_{23})$  is simply determined by the expression:

$$\tan(\theta_{23}) = \left| \frac{\cot\left((\phi_l - \phi_\nu)\frac{j}{2}\right)}{\cos(\theta_l)} \right| = \left| \frac{\cot\left(\frac{\pi(m_l - m_\nu)j}{n}\right)}{\cos(\theta_l)} \right| \quad (19)$$

Taking the argument of the cotangent to be either  $\frac{2\pi}{7}$  or  $\frac{3\pi}{14}$  leads to the numerical values

$$\tan(\theta_{23}) \approx 0.7975 \left| \frac{1}{\cos(\theta_l)} \right| \quad \text{or} \quad \tan(\theta_{23}) \approx 1.254 \left| \frac{1}{\cos(\theta_l)} \right|$$

At the same time the sine of  $\theta_l$  is determined by the (13) element of  $V_{MNS}$ , i.e. by the value of  $\sin(\theta_{13})$ :  $|(V_{MNS}^{23})_{13}| = \left| \sin\left((\phi_l - \phi_\nu)\frac{j}{2}\right) \sin(\theta_l) \right| = \left| \sin\left(\frac{\pi(m_l - m_\nu)j}{n}\right) \sin(\theta_l) \right|$ , which gives for  $\frac{2\pi}{7}$  and  $\frac{3\pi}{14}$   $|(V_{MNS}^{23})_{13}| = 0.7818 |\sin(\theta_l)|$  and  $0.6235 |\sin(\theta_l)|$ , respectively, i.e.  $|\sin(\theta_l)|$  has to be as small as possible to fulfill the experimental bound on  $\sin^2(\theta_{13})$ . Then  $|\cos(\theta_l)| \approx 1$  holds so that we can deduce the approximate values 0.7975 and 1.254 for  $\tan(\theta_{23})$  from Eq.(19). These correspond to  $\theta_{23} \approx 38.57^\circ$  ( $\sin^2(\theta_{23}) \approx 0.3888$ ) and  $\theta_{23} \approx 51.43^\circ$  ( $\sin^2(\theta_{23}) \approx 0.6113$ ), i.e. they are at the boundaries of the  $2\sigma$  range for  $\sin^2(\theta_{23})$  [12]. Similar statements hold in case of  $|V_{MNS}^{33}|$ .

Furthermore, one observes that in all cases the CP phase  $\delta$  is trivial, i.e. 0 or  $\pi$  with a numerical precision of  $\mathcal{O}(10^{-6})$ . Therefore  $J_{CP}$  always vanishes. In order to understand this result, we have a look at the formulae given for  $V_{mix}^{21}$ ,  $V_{mix}^{22}$ ,  $V_{mix}^{31}$  and  $V_{mix}^{32}$  in Appendix A. As a common feature the (13) element of the mixing matrix is given by

$$\frac{1}{2} [-(1 + e^{-i(\phi_l - \phi_\nu)j}) \sin(\theta_l) \cos(\theta_\nu) + 2e^{i\alpha} \cos(\theta_l) \sin(\theta_\nu)] \quad (20)$$

In all cases,  $\theta_l$  and  $\theta_\nu$  are predominantly determined by one element of the first row and the third column of  $V_{MNS}$ , respectively. Then  $\alpha$  can be used in order to minimize the absolute value of

Element	Cosine	$\theta_l$	$\theta_\nu$	$\alpha$	$\sin^2(\theta_{12})$	$\sin^2(\theta_{23})$	$\sin^2(\theta_{13})$	$\delta$
(21)	$\cos(\frac{3\pi}{7})$	0.9790	0.7881	4.937	0.2957	0.5085	$7.037 \times 10^{-2}$	$\sim \pi$
	$\cos(\frac{5\pi}{14})$	1.1829	0.6725	5.161	0.3001	0.4999	$6.173 \times 10^{-3}$	$\sim 0$
(22)	$\cos(\frac{5\pi}{14})$	—	—	—	—	—	—	—
	$\cos(\frac{2\pi}{7})$	0.7728	0.4486	5.386	0.2999	0.4996	$6.668 \times 10^{-3}$	$\sim \pi$
(23)	$\cos(\frac{2\pi}{7})$	—	—	—	—	—	—	—
	$\cos(\frac{3\pi}{14})$	—	—	—	—	—	—	—
(31)	$\cos(\frac{3\pi}{7})$	0.9790	0.7881	4.937	0.2957	0.4915	$7.037 \times 10^{-2}$	$\sim 0$
	$\cos(\frac{5\pi}{14})$	1.1829	0.6725	5.161	0.3001	0.5001	$6.173 \times 10^{-3}$	$\sim \pi$
(32)	$\cos(\frac{2\pi}{7})$	0.7728	0.4486	5.386	0.2999	0.5004	$6.668 \times 10^{-3}$	$\sim 0$
(33)	$\cos(\frac{2\pi}{7})$	—	—	—	—	—	—	—
	$\cos(\frac{3\pi}{14})$	—	—	—	—	—	—	—

Table 3: Numerical results for  $V_{MNS}$  in case of  $\sin^2(\theta_{13}) \leq 0.1$  and 10% errors for the other two sine squares.  $\delta$  is given with a precision of  $\mathcal{O}(10^{-6})$ .

the (13) element of  $V_{MNS}$ . A minimization with respect to  $\alpha$  shows

$$\alpha = -(\phi_l - \phi_\nu) \frac{j}{2} + \pi y = -\frac{\pi}{n} (m_l - m_\nu) j + \pi y \quad \text{with } y \in \mathbb{Z}_0 \quad (21)$$

The minimum value for  $|\sin(\theta_{13})|$  is then  $|\cos((\phi_l - \phi_\nu) \frac{j}{2}) \sin(\theta_l) \cos(\theta_\nu) + (-1)^{y+1} \cos(\theta_l) \sin(\theta_\nu)|$ . However, in all cases the expression is only minimized for  $y = 0, 2, \dots$ , as the involved sines and cosines are all positive (remember that  $\theta_l$  and  $\theta_\nu$  are restricted to be smaller than  $\frac{\pi}{2}$  by definition and also  $(\phi_l - \phi_\nu) \frac{j}{2} = \frac{\pi}{n} (m_l - m_\nu) j$  which is the argument of the cosine displayed in the tables is always smaller than  $\frac{\pi}{2}$ ). As  $J_{CP}$  is proportional to  $\sin((\phi_l - \phi_\nu) \frac{j}{2} + \alpha)$ , it is zero for the calculated value of  $\alpha$ . Therefore  $\delta$  must be either 0 or  $\pi$ . Additionally, we found an explanation for the values of  $\alpha$  shown in Table 3 given in terms of the group theoretical quantities, i.e.  $2\pi - \frac{3\pi}{7} \approx 4.937$ ,  $2\pi - \frac{5\pi}{14} \approx 5.161$  and  $2\pi - \frac{2\pi}{7} \approx 5.386$ . Since  $\alpha$  has to lie in  $[0, 2\pi)$ ,  $y$  equals two in all cases, see Eq.(21).

As a last observation we report that there exist similarities among the different cases, e.g. fixing the (21) element to be  $\cos(\frac{3\pi}{7})$  is similar to fixing the (31) element to the same value. In both cases the fit values of  $\theta_l$ ,  $\theta_\nu$  and  $\alpha$  are the same. Therefore, the results for  $\sin^2(\theta_{12})$  and  $\sin^2(\theta_{13})$  coincide (up to  $\mathcal{O}(10^{-6})$ ), while  $\sin^2(\theta_{23})$  is shifted from being  $0.5 + \epsilon$  to  $0.5 - \epsilon$  with  $\epsilon \approx 0.0085$  and the CP phase  $\delta$  shifts from  $\pi$  to 0. Looking at the mixing matrices one recognizes that these similarities are due to the fact that the second and the third row are interchanged.

Using the  $2\sigma$  bound  $\sin^2(\theta_{13}) \leq 0.025$  no solution with  $\chi^2 < 1$  is found in the cases in which the (21) or the (31) element is fixed to the value  $\cos(\frac{3\pi}{7})$ , since the values for  $\sin^2(\theta_{13})$  shown in Table 3 are quite large. For the other configurations we again find viable fits in which the values  $\theta_l$ ,  $\theta_\nu$  and  $\alpha$  are very similar to the ones given in Table 3.

Until now we only investigated the cases in which the group theoretically fixed element is given by one of the cosines shown in Table 2. However, as already remarked several times we can also look at cases in which the cosine is for example  $\cos(\frac{4\pi}{7})$  instead of  $\cos(\frac{3\pi}{7})$ , since  $\cos(\frac{4\pi}{7})$  is just the negative of  $\cos(\frac{3\pi}{7})$ . In terms of group theoretical quantities this corresponds to sending  $m_l$

Element ( $ij$ )	Possible cosines
(11)	$\cos(\frac{3\pi}{14}) (\approx 0.7818)$
(12)	$\cos(\frac{2\pi}{7}) (\approx 0.6235)$
(21)	$\cos(\frac{5\pi}{14}) (\approx 0.4339)$
(22)	$\cos(\frac{2\pi}{7}) (\approx 0.6235)$
(31)	$\cos(\frac{5\pi}{14}) (\approx 0.4339)$
(32)	$\cos(\frac{2\pi}{7}) (\approx 0.6235)$

Table 4: Possibilities for the group theoretically determined element in  $V_{MNS}$ , if TBM is assumed to be the best fit. For further conventions, see Table 2.

to  $n - m_l$  and therefore  $\phi_l$  to  $2\pi - \phi_l$ <sup>14</sup>. The general forms of the mixing matrices given in Appendix A show that such a transformation does not change the absolute values of the matrix elements, if we replace the phase  $\alpha$  by  $-\alpha$  ( $2\pi - \alpha$ ) at the same time. In contrast to this  $J_{CP}$  is not invariant and changes its sign. In the analysis of the leptonic mixing parameters this is not relevant, since the phase(s) have not been measured. Moreover, in all cases considered here  $J_{CP}$  is almost zero (up to  $\mathcal{O}(10^{-6})$ ). Therefore, we get the same results for these equivalent cases. Note, that in case of the quark mixing matrix we would have to expect different results, since there  $J_{CP}$  is known from experiment and its sign change leads to a distinct solution.

Apart from studying how well one can accommodate the experimentally allowed ranges, it is also interesting to see whether one can reproduce some special mixing pattern in the lepton sector. In the following we discuss the TBM scenario which has initially been discussed in [13], since all elements of the lepton mixing matrix can be written in terms of fractions of square roots  $\frac{1}{\sqrt{2}}$ ,  $\frac{1}{\sqrt{3}}$  and  $\frac{1}{\sqrt{6}}$ :

$$V_{MNS}^{TBM} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (22)$$

corresponding to sines of the mixing angles:

$$\sin^2(\theta_{23}^{TBM}) = \frac{1}{2}, \quad \sin^2(\theta_{12}^{TBM}) = \frac{1}{3} \quad \text{and} \quad \sin^2(\theta_{13}^{TBM}) = 0.$$

However, it turned out to be not just an assumption of a special form of  $V_{MNS}$ , but it is a robust outcome of certain models based on the discrete non-abelian symmetries  $A_4$  or  $T'$  [2, 3]. Therefore we want to analyze whether we can also accommodate the TBM with mixing matrices of the form  $V_{mix}$  as given in Appendix A. The uncertainty in the mixing matrix elements is taken to be 10%, i.e. the fixed element given by cosine  $|\cos(\frac{l\pi}{7})|$  for  $l = 0, 1, 2, \dots, 6$  or  $|\cos(\frac{l\pi}{14})|$  with  $l = 0, 1, 2, \dots, 13$  should lie in one of the ranges:

$$V_{MNS}^{TBM}(\text{range}) = \begin{pmatrix} 0.73 - 0.90 & 0.52 - 0.64 & < 0.20 \\ 0.37 - 0.45 & 0.52 - 0.64 & 0.64 - 0.78 \\ 0.37 - 0.45 & 0.52 - 0.64 & 0.64 - 0.78 \end{pmatrix} \quad (23)$$

The bound on the (13) element is taken to be the same as in Eq.(18). As shown in Table 4, the elements (11) and (12) can now be described by a cosine of the announced form, while we find

<sup>14</sup>Thereby, we have set  $\phi_\nu$  to zero and  $j$  to 1 without loss of generality.

Element	Cosine	$\theta_l$	$\theta_\nu$	$\alpha$	$\sin^2(\theta_{12})$	$\sin^2(\theta_{23})$	$\sin^2(\theta_{13})$	$\delta$
(11)	$\cos(\frac{3\pi}{14})$	$0.4396 - 1.131$	1.139	$\in [0, 2\pi)$	0.3441	0.5000	$6.808 \times 10^{-2}$	$\in [\sim 0, \sim 2\pi)$
(12)	$\cos(\frac{2\pi}{7})$	—	—	—	—	—	—	—
(21)	$\cos(\frac{5\pi}{14})$	1.132	0.6697	5.161	0.3334	0.5000	$1.968 \times 10^{-3}$	$\sim 0$
(22)	$\cos(\frac{2\pi}{7})$	0.8235	0.4557	5.386	0.3331	0.4991	$1.245 \times 10^{-2}$	$\sim \pi$
(31)	$\cos(\frac{5\pi}{14})$	1.132	0.6697	5.161	0.3334	0.5000	$1.968 \times 10^{-3}$	$\sim \pi$
(32)	$\cos(\frac{2\pi}{7})$	0.8235	0.4557	5.386	0.3331	0.5009	$1.245 \times 10^{-2}$	$\sim 0$

Table 5: Numerical results in the case of TBM. We assume that the bound on  $\sin^2(\theta_{13})$  is 0.1 and 10% errors for the other two sine squares. The values of  $\delta$  have a numerical precision of  $\mathcal{O}(10^{-6})$ . Note that in case of the (11) element being  $\cos(\frac{3\pi}{14})$   $\delta$  can take arbitrary values. (for details see text).

less possibilities for the other elements compared to the case of the experimentally allowed range, see Table 2. This analysis is analogous to the one above. Again, we display the results for the fits using the loose bound for  $\sin^2(\theta_{13})$ ,  $\sin^2(\theta_{13}) \leq 0.1$ . Similar to above, there is a case in which we have not found a fit with  $\chi^2 < 1$ . For the cases in which either the (21), (22), (31) or (32) element is determined by group theory all statements made above can also be applied here, i.e. the CP phase  $\delta$  is either 0 or  $\pi$ , the phase  $\alpha$  is fixed to a certain value which minimizes  $|\sin(\theta_{13})|$  and there exists a similarity among the cases with a fixed (21) ((22)) element and a fixed (31) ((32)) element. Therefore, we focus on the discussion of a group theoretically determined (11) element of  $V_{MNS}$ . This case exhibits some new features not present in the other ones. First of all, we find that  $\theta_l$  can take values in a certain range instead of being fixed to a single value. All of them lead to the same mixing angles. The same is true for  $\alpha$  which varies between 0 and  $2\pi$ . This is related to the fact that we do not fit the CP phase  $\delta$  (or equivalently the Jarlskog invariant  $J_{CP}$ ). As a result  $J_{CP}$  can take any value in the range  $(-5.776...5.776) \times 10^{-2}$ . We observe that  $\theta_\nu$  is fixed by the fit of  $\sin^2(\theta_{12})$  and  $\sin^2(\theta_{13})$ . Fitting them at the same time leads, unfortunately, to a too large value for  $\sin^2(\theta_{13})$  (see Table 5). The allowed range for  $\theta_l$  can then be found analytically under the assumption that  $\sin^2(\theta_{23}) = \frac{1}{2}$ , since in this case the (23) and (33) element of  $V_{MNS}$  have to be equal. Equating the expressions  $|(V_{mix}^{11})_{23}|^2$  and  $|(V_{mix}^{11})_{33}|^2$  found in Appendix A leads to

$$\tan(2\theta_l) = \frac{\sin^2(\theta_\nu) - \cos^2((\phi_l - \phi_\nu)\frac{i}{2}) \cos^2(\theta_\nu)}{\cos((\phi_l - \phi_\nu)\frac{i}{2} + \alpha) \cos((\phi_l - \phi_\nu)\frac{i}{2}) \sin(2\theta_\nu)} \quad (24)$$

with  $\theta_\nu$  determined by  $\sin^2(\theta_{12,13})$ . Allowing  $\alpha \in [0, 2\pi)$  one finds the maximal range of  $\theta_l$  to be  $z \leq \theta_l \leq \frac{\pi}{2} - z$  with  $z \approx 0.4396$  for  $\theta_\nu \approx 1.139$  and  $(\phi_l - \phi_\nu)\frac{i}{2} = \frac{3\pi}{14}$  which corresponds to the numerical values given in Table 5. Furthermore, Eq.(24) shows that  $\theta_l$  is a function of  $\alpha$ . Demanding  $\sin^2(\theta_{13}) \leq 0.025$  removes the possibility that the (11) element of  $V_{MNS}$  is determined by group theory, while it leads to expected slight changes in the results of the fits for the rest of the cases. As expected, a comparison of these results to the ones for the experimental best fit values shows that there are only small changes in the precise values of  $\theta_l$ ,  $\theta_\nu$  and  $\alpha$ .

In this section we have shown that it is possible to fit the lepton mixing angles [12] in a framework in which one of the elements of  $V_{MNS}$  is completely determined by group theoretical quantities of a dihedral flavor symmetry. We restricted ourselves to the dihedral groups  $D_7$  and  $D_{14}$ , since they allow us to explain the Cabibbo angle via group theory. A main result of the analysis is that  $J_{CP}$

vanishes in all cases. Using the formulae given in Appendix A one can show that the vanishing of  $J_{CP}$  is correlated with the minimization of  $|\sin(\theta_{13})|$ . As the bound on  $|\sin(\theta_{13})|$  is the strongest constraint on the solution, we showed our results for a very loose bound. Furthermore, we analyzed how well one can mimic the TBM scenario. We found several possible solutions. One of these is of special interest, since it also allows for non-trivial CP violation. However, the corresponding value of  $\sin(\theta_{13})$  is very large and therefore this solution is disfavored. These results demonstrate that it is possible to treat the lepton mixings in the same way as the ones of the quarks. Small corrections are expected in a complete model, e.g. due to explicit breakings of the preserved subgroups.

## 8 Summary and Conclusions

In [1] we studied the dihedral groups as possible flavor symmetries. The key feature there is the fact that the flavor symmetry is not broken in an arbitrary way, but one requires that a subgroup has to be preserved in all cases. It turned out that the number of possible mass matrix structures which arise from such a setup is very limited, if we assume that the mass matrix has a non-vanishing determinant. As a first application we discussed in [1] the possibility to describe one element of the CKM mixing matrix only in terms of group theoretical quantities, i.e. the index  $n$  of the dihedral group  $D_n$ , the index  $j$  of the representation  $\mathbf{2}_j$  under which the fermions transform and the indices  $m_u$  and  $m_d$  of the residual subgroups  $Z_2 = \langle BA^{m_u} \rangle$  and  $Z_2 = \langle BA^{m_d} \rangle$ :

$$\frac{1}{2} \left| 1 + e^{i(\phi_u - \phi_d)j} \right| = \left| \cos\left((\phi_u - \phi_d) \frac{j}{2}\right) \right| = \left| \cos\left(\frac{\pi}{n} (m_u - m_d)j\right) \right| \quad (25)$$

where  $\phi_u = \frac{2\pi}{n} m_u$  and  $\phi_d = \frac{2\pi}{n} m_d$ . Eq.(25) shows that a non-trivial mixing angle demands  $m_u \neq m_d$ , i.e. the two  $Z_2$  subgroups have to be distinct. It has been pointed out that  $|V_{us}|$  can be fitted well with  $\cos(\frac{3\pi}{7}) \approx 0.2225$ . In this work, we first studied which of the other elements of  $V_{CKM}$  can also be approximated well by Eq.(25) for certain values of the group index  $n$ . For the smallest two appropriate values of  $n$ ,  $n = 7$  and  $n = 14$ , each element of the  $1 - 2$  sub-block of  $V_{CKM}$  can be put into this form, i.e.  $|V_{ud}| \approx |V_{cs}| \approx \cos(\frac{\pi}{14})$  and  $|V_{us}| \approx |V_{cd}| \approx \cos(\frac{3\pi}{7})$ . A numerical analysis showed that the other two mixing angles,  $\theta_{13}^q$  and  $\theta_{23}^q$ , and the CP phase  $\delta$  can be fitted well with the free angles  $\theta_{u,d}$  and the phase  $\alpha = \beta_u - \beta_d$ . Since the fixed element cannot be fitted, the results for  $V_{CKM}$  are very close to the experimental values, but not within the (very small) experimental errors [5]. However, several sources of corrections exist in a complete model, e.g. possible higher-dimensional operators as well as small, but explicit, breakings of the preserved subgroups. In a next step, we presented a low energy model for the quark sector which incorporates the described idea. The flavor symmetry is taken to be  $D_7$ . It is broken only spontaneously at the electroweak scale by Higgs fields transforming as doublets under  $SU(2)_L$ . With a numerical fit we showed that all quark masses and mixing parameters can be fitted well at the same time. As the VEV configuration determines the subgroup to which the flavor symmetry is broken, it is necessary to investigate whether this can be achieved by the Higgs potential. We studied this issue for the minimal model in which the three Higgs fields  $H_s^u$  and  $H_{1,2}^u$  couple to up quarks, while the fields  $H_s^d$  and  $H_{1,2}^d$  couple to down quarks only. Unfortunately, the Higgs potential containing only  $H_s^{d,u}$  and  $H_{1,2}^{d,u}$  has an accidental symmetry which is necessarily broken by the desired VEV configuration. For this reason we had to add two further Higgs fields to the model. These do not directly couple to the fermions due to their  $D_7$  transformation properties. The couplings of  $H_s^{d,u}$  and  $H_{1,2}^{d,u}$  to these Higgs fields break all accidental symmetries of the potential. A numerical study showed that the needed VEV configuration can be achieved

with this potential. However, there are two obstacles: first of all if the quartic couplings of the Higgs potential are in the perturbative range and the mass parameters are taken to be around the electroweak scale, the Higgs masses turn out to be too small, i.e. some of them are even below the LEP bound [14]. This could be cured by adding mass terms which break the flavor symmetry softly in the Higgs potential and allow for larger Higgs masses. However, even then this model might suffer from the problem that FCNCs induced by the additional Higgs fields are too large to pass the experimental bounds. The second obstacle is the fact that we are only able to accommodate the VEV configuration as one possible solution of the Higgs potential, but not as a favored solution. Moreover, there is in general no way to stabilize such a configuration against further corrections in a multi-Higgs doublet potential. Therefore this model is meant as a proof of principle rather than a realistic model. A way to circumvent these problems is to disentangle the scales of the electroweak and the flavor symmetry breaking by using flavored gauge singlets instead of Higgs doublets and thereby break the dihedral symmetry at higher energies [4].

Accounting for the fact that the Cabibbo angle  $\theta_C$  is roughly an order of magnitude larger than the two other mixing angles  $\theta_{13}^q$  and  $\theta_{23}^q$  one can look for models in which  $\theta_C$  is given in terms of group theoretical quantities and  $\theta_{13}^q$  and  $\theta_{23}^q$  vanish at LO. As shown in Section 6 this can be implemented successfully in at least two different ways: *a.*) one can simply reduce the number of Higgs fields in the model by omitting some fields which are allowed to have a non-trivial VEV in principle; *b.*) one can break the dihedral symmetry down to one of its dihedral subgroups,  $D_q$ ,  $q > 1$ , instead of  $Z_2$ . Case *a.*) has the slight disadvantage that the resulting mass matrices now also depend on the choice of the scalar fields and are not only determined by the representations under which the fermions transform and the group theory of the dihedral symmetry. Case *b.*) on the other hand does not suffer from this sort of arbitrariness. However, it cannot be realized with all dihedral symmetries, since not all of them have dihedral subgroups  $D_q$  with  $q > 1$ . The group  $D_7$  which has been used in this paper only has  $Z_2$  and  $Z_7$  as subgroups, since its group index is prime. Therefore in the shown examples (for case *b.*) the flavor symmetry is taken to be  $D_{14}$  instead. The preserved subgroups are of the form  $D_2 = \langle A^7, B A^m \rangle$ . Also here it is necessary to break down to two different  $D_2$  groups in the up quark and down quark sector in order to generate a non-vanishing Cabibbo angle. One possible choice is  $m_u = 6$  and  $m_d = 0$ .

Finally, we also studied the lepton mixing matrix  $V_{MNS}$  numerically. In order to apply the results of the mixing matrices found in Section 2 we restricted ourselves to the discussion of Dirac neutrinos with a normally ordered spectrum, i.e. the neutrinos have the same properties as the other fermions. Since the elements of  $V_{MNS}$  are much less constrained than the ones of  $V_{CKM}$  much more combinations of the group theoretical quantities  $n$ ,  $j$ ,  $m_l$  and  $m_\nu$  can be used in order to describe an element of  $V_{MNS}$ . However, since we expect that the leptons transform under the same flavor symmetry as the quarks, we only considered cases in which the group index  $n$  is fixed to  $n = 7$  or  $n = 14$ . A numerical analysis shows that the experimental fit values of the mixing angles can be accommodated well in most of the cases. The strongest constraint seems to arise from the upper bound on the reactor mixing angle  $\theta_{13}$ . Therefore we performed fits with two different bounds on  $\sin^2(\theta_{13})$ . The results which are shown in Section 7 correspond to a very loose bound,  $\sin^2(\theta_{13}) \leq 0.1$  (which exceeds the  $4\sigma$  bound [12]), while in the other fit the  $2\sigma$  bound,  $\sin^2(\theta_{13}) \leq 0.025$ , has been used. A common feature of all fits is the fact that  $J_{CP}$  vanishes. As shown in Section 7 the condition which minimizes the (13) element of  $V_{MNS}$  whose absolute value is  $\sin(\theta_{13})$  also implies  $J_{CP} = 0$ . Furthermore, it turns out that the fit parameters  $\theta_l$ ,  $\theta_\nu$  and  $\alpha$  are fixed to a unique value in all cases. In addition to this, we also studied how well one could mimic the TBM scenario with a mixing matrix resulting from a dihedral flavor symmetry with preserved subgroups. Again, we only considered the cases  $n = 7$  and  $n = 14$ . Our results are similar to the ones found in case of the fit to the experimental best

fit values, i.e. a successful fit is possible in several cases. Similar to above, the main restriction seems to come from the requirement to pass the bound on  $\sin^2(\theta_{13})$ . For this reason again two different bounds on  $\sin^2(\theta_{13})$  have been used. Apart from the cases which lead to similar results as above, we observe one additional case, namely if  $(V_{MNS})_{11}$  is fixed to be  $\cos(\frac{3\pi}{14}) \approx 0.7818$ . Unlike in the other cases we can observe CP violation here, i.e.  $J_{CP}$  can take any value between  $-5.776 \times 10^{-2}$  and  $5.776 \times 10^{-2}$ . In contrast to this, the results of the fit of the mixing angles do not vary. Unfortunately,  $\sin^2(\theta_{13})$  is very large and therefore a model incorporating this solution is disfavored, if contributions from, for example, higher-dimensional operators are not able to lower  $\sin^2(\theta_{13})$ .

In the whole discussion we focussed on the case of Dirac neutrinos, since then all formulae found in case of the quarks are applicable also to the lepton sector. However, in general neutrinos can be Majorana particles. If we assume that they acquire masses from Higgs triplets only, i.e. there are no right-handed neutrinos, the possible matrix structures are  $M_5$  (Eq.(5)) and a block structure (Eq.(3)), see Section 2. Compared to the case of Dirac neutrinos the mixing matrix  $U_\nu$  is now determined by  $U_\nu^\dagger M_\nu U_\nu^* = \text{diag}(m_1, m_2, m_3)$  and therefore in general contains Majorana phases. This, however, does not matter for the analysis done in Section 7, since there only the absolute values of the matrix elements are relevant. Things can change, if we consider the type 1 seesaw instead. As we then deal with the Dirac neutrino mass matrix and the right-handed Majorana mass matrix, these mass matrices can preserve different subgroups of the flavor symmetry. This is, for example, the case in the models [15, 16] by Grimus and Lavoura<sup>15</sup>. The situation can be even more complicated, if the model also includes Higgs triplets. Then all these three matrices, i.e. the Majorana mass matrix of the left-handed neutrinos, the one of the right-handed ones and the Dirac neutrino mass matrix, can conserve different subgroups of the original symmetry and in general no definite statements can be made about the resulting mixing matrix. Furthermore we assumed throughout our analysis that the neutrinos have the same mass ordering as all the other fermions. However, due to the unknown sign of the atmospheric mass squared difference it is also possible that the neutrino mass hierarchy is inverted ( $m_3 < m_1 < m_2$ ).

Our study is by no means a complete study of all possible mixing structures which can in principle arise from a dihedral flavor symmetry with preserved subgroups. For example, in all cases we presented here the subgroups which are preserved in the up and down quark sector have the same group structure, i.e. they are either both  $Z_2$  or  $D_2$  groups. In general, however, these group structures could be different. Successful examples which employ subgroups of different structures are the  $A_4$  ( $T'$ ) models as well as the models by Grimus and Lavoura. As already mentioned in the Introduction, in the  $A_4$  ( $T'$ ) model the conserved subgroups are  $Z_2$  ( $Z_4$ ) and  $Z_3$  in order to predict TBM in the lepton sector. In the first model [15] by Grimus and Lavoura the flavor symmetry  $D_4 \times Z_2^{(aux)}$  is broken either to  $D_2$ ,  $Z_2$  or is left intact (see [1]). Similarly, in their second model [16] with  $D_3 \times Z_2^{(aux)}$  as flavor symmetry  $Z_3$ ,  $Z_2$  and  $D_3$  are the preserved subgroups (see also [1]). Both models lead to vanishing  $\theta_{13}$  and maximal atmospheric mixing (for the leptons). This shows that the usage of subgroups of different group structure leaves much more possibilities than the ones shown here. As the complete study of mass matrix structures (with  $\det(M) \neq 0$ ) which can arise from a dihedral symmetry, if a subgroup remains preserved, already exists [1], it is only the question how to combine these results in order to get further interesting predictions for the mixing patterns of quarks and leptons. One interesting example, namely the explanation of the Cabibbo angle, has been studied in detail in this work.

Finally, let us remark that a common feature of the model(s) shown here and the successful

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<sup>15</sup>In their models the Dirac neutrino mass matrix is actually invariant under the whole dihedral symmetry, i.e. stems solely from VEVs of fields which transform trivially under the flavor group.



$A_4$  ( $T'$ ) models is the need for an additional  $Z_n^{(aux)}$  symmetry which can separate the different sectors according to the different subgroups of the flavor symmetry which should be preserved. Due to such an additional symmetry an embedding of these models into an  $SO(10)$  GUT is in general not straightforward. However, assigning the quarks to

$$Q_1, u_1^c \sim (\underline{\mathbf{1}}_1, +1), \quad \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix}, \begin{pmatrix} u_2^c \\ u_3^c \end{pmatrix} \sim (\underline{\mathbf{2}}_1, +1), \quad d_1^c \sim (\underline{\mathbf{1}}_1, -1), \quad \begin{pmatrix} d_2^c \\ d_3^c \end{pmatrix} \sim (\underline{\mathbf{2}}_1, -1) \quad (26)$$

under  $D_7 \times Z_2^{(aux)}$  as done in Section 4.1.2 still allows an embedding into  $SU(5)$  multiplets.

*Note added:* At the final stages of this work the paper [17] by C. S. Lam appeared. He also deals with the fact that non-trivial subgroups of some discrete flavor symmetry can help to explain a certain mixing pattern and also very briefly mentions that the Cabibbo angle might be the result of some dihedral group.

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## A Possible Forms of $V_{mix}$

According to the three possible identifications of the eigenvalue  $c - d$  there exist three possible diagonalization matrices in each sector (up and down sector, charged lepton and neutrino sector)  $U$ ,  $U'$  and  $U''$  which are shown in Section 3.1. Out of these one can form nine possible mixing matrices  $V_{mix}^{ab} = W_1^T W_2^*$  with  $a, b = 1, 2, 3$  and  $W_i \in \{U, U', U''\}$  where  $W_i$  depends on the group theoretical phase  $\phi_i$  (the index  $m_i$ ) and contains the parameters  $\theta_i$  and  $\beta_i$ . They all have the property that one of their matrix elements, namely the element  $(ab)$ , is completely determined by group theory, i.e. by the index  $n$  of the dihedral group, by the index  $j$  of the two-dimensional representation  $\underline{2}_j$  under which two of three generations of  $SU(2)_L$  doublets transform and by the breaking directions described by  $m_1$  and  $m_2$  in the two different sectors. In the following we abbreviate  $\beta_1 - \beta_2$  with  $\alpha$ ,  $\sin(\theta_i)$  with  $s_i$  and  $\cos(\theta_i)$  with  $c_i$ .

$$\begin{aligned}
V_{mix}^{11} &= \frac{1}{2} \begin{pmatrix} 1 + e^{i(\phi_1 - \phi_2)j} & (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 \\ -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 & (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 \\ (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 & -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 \end{pmatrix} \\
V_{mix}^{12} &= \frac{1}{2} \begin{pmatrix} (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & 1 + e^{i(\phi_1 - \phi_2)j} & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 \\ (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 \\ -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 \end{pmatrix} \\
V_{mix}^{13} &= \frac{1}{2} \begin{pmatrix} (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 & 1 + e^{i(\phi_1 - \phi_2)j} \\ (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 & -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 \\ -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 & (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 \end{pmatrix} \\
V_{mix}^{21} &= \frac{1}{2} \begin{pmatrix} -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 & (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 \\ 1 + e^{i(\phi_1 - \phi_2)j} & (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 \\ (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 & -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 \end{pmatrix} \\
V_{mix}^{22} &= \frac{1}{2} \begin{pmatrix} (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 \\ (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & 1 + e^{i(\phi_1 - \phi_2)j} & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 \\ -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 \end{pmatrix} \\
V_{mix}^{23} &= \frac{1}{2} \begin{pmatrix} (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 & -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 \\ (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 & 1 + e^{i(\phi_1 - \phi_2)j} \\ -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 & (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 \end{pmatrix} \\
V_{mix}^{31} &= \frac{1}{2} \begin{pmatrix} -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 & (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 \\ (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 & -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 \\ 1 + e^{i(\phi_1 - \phi_2)j} & (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 \end{pmatrix} \\
V_{mix}^{32} &= \frac{1}{2} \begin{pmatrix} (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 \\ -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 \\ (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & 1 + e^{i(\phi_1 - \phi_2)j} & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 \end{pmatrix} \\
V_{mix}^{33} &= \frac{1}{2} \begin{pmatrix} (1 + e^{-i(\phi_1 - \phi_2)j}) s_1 s_2 + 2e^{i\alpha} c_1 c_2 & -(1 + e^{-i(\phi_1 - \phi_2)j}) s_1 c_2 + 2e^{i\alpha} c_1 s_2 & -(e^{-i\phi_1 j} - e^{-i\phi_2 j}) s_1 \\ -(1 + e^{-i(\phi_1 - \phi_2)j}) c_1 s_2 + 2e^{i\alpha} s_1 c_2 & (1 + e^{-i(\phi_1 - \phi_2)j}) c_1 c_2 + 2e^{i\alpha} s_1 s_2 & (e^{-i\phi_1 j} - e^{-i\phi_2 j}) c_1 \\ (e^{i\phi_1 j} - e^{i\phi_2 j}) s_2 & -(e^{i\phi_1 j} - e^{i\phi_2 j}) c_2 & 1 + e^{i(\phi_1 - \phi_2)j} \end{pmatrix}
\end{aligned}$$

The measure of CP-violation  $J_{CP}^{ab}$  is given for the matrices  $V_{mix}^{ab}$  as

$$J_{CP}^{11} = J_{CP}(j, \phi_1, \phi_2; \theta_1, \theta_2, \alpha), \quad J_{CP}^{12} = -J_{CP}(j, \phi_1, \phi_2; \theta_1, \theta_2, \alpha), \quad J_{CP}^{13} = J_{CP}(j, \phi_1, \phi_2; \theta_1, \theta_2, \alpha) \quad (27)$$

$$J_{CP}^{21} = -J_{CP}(j, \phi_1, \phi_2; \theta_1, \theta_2, \alpha), \quad J_{CP}^{22} = J_{CP}(j, \phi_1, \phi_2; \theta_1, \theta_2, \alpha), \quad J_{CP}^{23} = -J_{CP}(j, \phi_1, \phi_2; \theta_1, \theta_2, \alpha) \quad (28)$$

$$J_{CP}^{31} = J_{CP}(j, \phi_1, \phi_2; \theta_1, \theta_2, \alpha), \quad J_{CP}^{32} = -J_{CP}(j, \phi_1, \phi_2; \theta_1, \theta_2, \alpha), \quad J_{CP}^{33} = J_{CP}(j, \phi_1, \phi_2; \theta_1, \theta_2, \alpha) \quad (29)$$

$$\text{with } J_{CP}(j, \phi_1, \phi_2; \theta_1, \theta_2, \alpha) = -\frac{1}{8} \sin((\phi_1 - \phi_2)j) \sin\left(\frac{1}{2}(\phi_1 - \phi_2)j\right) \sin(2\theta_1) \sin(2\theta_2) \sin\left(\frac{1}{2}(\phi_1 - \phi_2)j + \alpha\right) \quad (30)$$

	classes				
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
G	$\mathbb{1}$	A	$A^2$	$A^3$	B
${}^\circ\mathcal{C}_i$	1	2	2	2	7
${}^\circ h_{\mathcal{C}_i}$	1	7	7	7	2
<u><b>1</b></u> <sub>1</sub>	1	1	1	1	1
<u><b>1</b></u> <sub>2</sub>	1	1	1	1	-1
<u><b>2</b></u> <sub>1</sub>	2	$2 \cos(\varphi)$	$2 \cos(2\varphi)$	$2 \cos(3\varphi)$	0
<u><b>2</b></u> <sub>2</sub>	2	$2 \cos(2\varphi)$	$2 \cos(4\varphi)$	$2 \cos(6\varphi)$	0
<u><b>2</b></u> <sub>3</sub>	2	$2 \cos(3\varphi)$	$2 \cos(6\varphi)$	$2 \cos(9\varphi)$	0

Table 6: Character table of the group  $D_7$ .  $\varphi$  is  $\frac{2\pi}{7}$ .  $\mathcal{C}_i$  are the classes of the group,  ${}^\circ\mathcal{C}_i$  is the order of the  $i^{\text{th}}$  class, i.e. the number of distinct elements contained in this class,  ${}^\circ h_{\mathcal{C}_i}$  is the order of the elements  $S$  in the class  $\mathcal{C}_i$ , i.e. the smallest integer ( $> 0$ ) for which the equation  $S^{{}^\circ h_{\mathcal{C}_i}} = \mathbb{1}$  holds. Furthermore the table contains one representative for each class  $\mathcal{C}_i$  given as product of the generators A and B of the group.

## B Group Theory of $D_7$

The group  $D_7$  has two one- and three two-dimensional irreducible representations which we denote as **1**<sub>1</sub>, **1**<sub>2</sub>, **2**<sub>1</sub>, **2**<sub>2</sub> and **2**<sub>3</sub>. **1**<sub>1</sub> is the trivial representation of the group. All two-dimensional representations are faithful. The order of the group is 14. The generator relations for the two generators A and B are:

$$A^7 = 1, \quad B^2 = 1, \quad ABA = B.$$

A and B can be chosen to be

$$\begin{aligned} A &= \begin{pmatrix} e^{\frac{2\pi i}{7}} & 0 \\ 0 & e^{-\frac{2\pi i}{7}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } \underline{\mathbf{2}}_1 \\ A &= \begin{pmatrix} e^{\frac{4\pi i}{7}} & 0 \\ 0 & e^{-\frac{4\pi i}{7}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } \underline{\mathbf{2}}_2 \\ A &= \begin{pmatrix} e^{\frac{6\pi i}{7}} & 0 \\ 0 & e^{-\frac{6\pi i}{7}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } \underline{\mathbf{2}}_3 \end{aligned}$$

For the one-dimensional representations **1**<sub>1</sub> and **1**<sub>2</sub> A and B can be found in the character table Table 6.

The Kronecker products are:

$$\begin{aligned} \underline{\mathbf{1}}_1 \times \mu &= \mu, \quad \underline{\mathbf{1}}_2 \times \underline{\mathbf{1}}_2 = \underline{\mathbf{1}}_1, \quad \underline{\mathbf{1}}_2 \times \underline{\mathbf{2}}_i = \underline{\mathbf{2}}_i \\ [\underline{\mathbf{2}}_1 \times \underline{\mathbf{2}}_1] &= \underline{\mathbf{1}}_1 + \underline{\mathbf{2}}_2, \quad \{\underline{\mathbf{2}}_1 \times \underline{\mathbf{2}}_1\} = \underline{\mathbf{1}}_2 \\ [\underline{\mathbf{2}}_2 \times \underline{\mathbf{2}}_2] &= \underline{\mathbf{1}}_1 + \underline{\mathbf{2}}_3, \quad \{\underline{\mathbf{2}}_2 \times \underline{\mathbf{2}}_2\} = \underline{\mathbf{1}}_2 \\ [\underline{\mathbf{2}}_3 \times \underline{\mathbf{2}}_3] &= \underline{\mathbf{1}}_1 + \underline{\mathbf{2}}_1, \quad \{\underline{\mathbf{2}}_3 \times \underline{\mathbf{2}}_3\} = \underline{\mathbf{1}}_2 \\ \underline{\mathbf{2}}_1 \times \underline{\mathbf{2}}_2 &= \underline{\mathbf{2}}_1 + \underline{\mathbf{2}}_3, \quad \underline{\mathbf{2}}_1 \times \underline{\mathbf{2}}_3 = \underline{\mathbf{2}}_2 + \underline{\mathbf{2}}_3, \\ \underline{\mathbf{2}}_2 \times \underline{\mathbf{2}}_3 &= \underline{\mathbf{2}}_1 + \underline{\mathbf{2}}_2, \end{aligned}$$

where  $\mu$  is any representation of the group and  $[\nu \times \nu]$  denotes the symmetric part of the product  $\nu \times \nu$ , while  $\{\nu \times \nu\}$  is the anti-symmetric one.

The Clebsch Gordan coefficients are trivial for  $\underline{1}_1 \times \mu$  and  $\underline{1}_2 \times \underline{1}_2$ . For  $\underline{1}_2 \times \underline{2}_i$  a non-trivial sign appears

$$\begin{pmatrix} B a_1 \\ -B a_2 \end{pmatrix} \sim \underline{2}_i$$

for  $B \sim \underline{1}_2$  and  $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{2}_i$ .  $\underline{1}_1$  and  $\underline{1}_2$  of  $\underline{2}_i \times \underline{2}_i$  are of the form

$$a_1 a'_2 + a_2 a'_1 \sim \underline{1}_1, \quad a_1 a'_2 - a_2 a'_1 \sim \underline{1}_2$$

for  $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} \sim \underline{2}_i$ . The two-dimensional representations also contained in these products read:

$$\text{for } i = 1 : \quad \begin{pmatrix} a_1 a'_1 \\ a_2 a'_2 \end{pmatrix} \sim \underline{2}_2$$

$$\text{for } i = 2 : \quad \begin{pmatrix} a_2 a'_2 \\ a_1 a'_1 \end{pmatrix} \sim \underline{2}_3$$

$$\text{for } i = 3 : \quad \begin{pmatrix} a_2 a'_2 \\ a_1 a'_1 \end{pmatrix} \sim \underline{2}_1$$

For the rest of the products  $\underline{2}_i \times \underline{2}_j$  we get:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{2}_1, \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sim \underline{2}_2 : \quad \begin{pmatrix} a_2 b_1 \\ a_1 b_2 \end{pmatrix} \sim \underline{2}_1, \quad \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix} \sim \underline{2}_3$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{2}_1, \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sim \underline{2}_3 : \quad \begin{pmatrix} a_2 b_1 \\ a_1 b_2 \end{pmatrix} \sim \underline{2}_2, \quad \begin{pmatrix} a_2 b_2 \\ a_1 b_1 \end{pmatrix} \sim \underline{2}_3$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{2}_2, \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sim \underline{2}_3 : \quad \begin{pmatrix} a_2 b_1 \\ a_1 b_2 \end{pmatrix} \sim \underline{2}_1, \quad \begin{pmatrix} a_2 b_2 \\ a_1 b_1 \end{pmatrix} \sim \underline{2}_2$$

All these formulae are just special cases of the more general formulae given in [1, 10] which hold for dihedral groups  $D_n$  with an arbitrary index  $n$ .

## C Higgs Potential

We begin by writing down the potential for the three Higgs fields  $H_s^u$  and  $H_{1,2}^u$ , which couple only to up quarks, i.e. are even under the additional  $Z_2^{(aux)}$  symmetry. The potential is of the same form as  $V_3$  in Eq.(14). As mentioned above, it has an accidental  $U(1)$  symmetry.

$$\begin{aligned}
V_u = & -(\mu_s^u)^2 H_s^{u\dagger} H_s^u - (\mu_D^u)^2 \left( \sum_{i=1}^2 H_i^{u\dagger} H_i^u \right) + \lambda_s^u (H_s^{u\dagger} H_s^u)^2 + \lambda_1^u \left( \sum_{i=1}^2 H_i^{u\dagger} H_i^u \right)^2 \\
& + \lambda_2^u (H_1^{u\dagger} H_1^u - H_2^{u\dagger} H_2^u)^2 + \lambda_3^u |H_1^{u\dagger} H_2^u|^2 \\
& + \sigma_1^u (H_s^{u\dagger} H_s^u) \left( \sum_{i=1}^2 H_i^{u\dagger} H_i^u \right) + \{ \sigma_2^u (H_s^{u\dagger} H_1^u) (H_s^{u\dagger} H_2^u) + \text{h.c.} \} + \sigma_3^u \left( \sum_{i=1}^2 |H_s^{u\dagger} H_i^u|^2 \right)
\end{aligned} \tag{31}$$

We have in addition five Higgs fields which are odd under the extra  $Z_2^{(aux)}$ . These are  $H_s^d$ ,  $H_{1,2}^d$  and  $\chi_{1,2}^d$ . The most general potential for these five scalar fields is

$$\begin{aligned}
V_d = & -(\mu_s^d)^2 H_s^{d\dagger} H_s^d - (\mu_D^d)^2 \left( \sum_{i=1}^2 H_i^{d\dagger} H_i^d \right) - (\tilde{\mu}_D^d)^2 \left( \sum_{i=1}^2 \chi_i^{d\dagger} \chi_i^d \right) \\
& + \lambda_s^d (H_s^{d\dagger} H_s^d)^2 + \lambda_1^d \left( \sum_{i=1}^2 H_i^{d\dagger} H_i^d \right)^2 + \tilde{\lambda}_1^d \left( \sum_{i=1}^2 \chi_i^{d\dagger} \chi_i^d \right)^2 + \lambda_2^d (H_1^{d\dagger} H_1^d - H_2^{d\dagger} H_2^d)^2 + \tilde{\lambda}_2^d (\chi_1^{d\dagger} \chi_1^d - \chi_2^{d\dagger} \chi_2^d)^2 \\
& + \lambda_3^d |H_1^{d\dagger} H_2^d|^2 + \tilde{\lambda}_3^d |\chi_1^{d\dagger} \chi_2^d|^2 + \sigma_1^d (H_s^{d\dagger} H_s^d) \left( \sum_{i=1}^2 H_i^{d\dagger} H_i^d \right) + \tilde{\sigma}_1^d (H_s^{d\dagger} H_s^d) \left( \sum_{i=1}^2 \chi_i^{d\dagger} \chi_i^d \right) \\
& + \{ \sigma_2^d (H_s^{d\dagger} H_1^d) (H_s^{d\dagger} H_2^d) + \text{h.c.} \} + \{ \tilde{\sigma}_2^d (H_s^{d\dagger} \chi_1^d) (H_s^{d\dagger} \chi_2^d) + \text{h.c.} \} + \sigma_3^d \left( \sum_{i=1}^2 |H_s^{d\dagger} H_i^d|^2 \right) + \tilde{\sigma}_3^d \left( \sum_{i=1}^2 |H_s^{d\dagger} \chi_i^d|^2 \right) \\
& + \tau_1^d \left( \sum_{i=1}^2 H_i^{d\dagger} H_i^d \right) \left( \sum_{i=1}^2 \chi_i^{d\dagger} \chi_i^d \right) + \tau_2^d (H_1^{d\dagger} H_1^d - H_2^{d\dagger} H_2^d) (\chi_1^{d\dagger} \chi_1^d - \chi_2^{d\dagger} \chi_2^d) \\
& + \{ \tau_3^d (H_1^{d\dagger} \chi_1^d) (H_2^{d\dagger} \chi_2^d) + \text{h.c.} \} + \tau_4^d \left( \sum_{i=1}^2 |H_i^{d\dagger} \chi_i^d|^2 \right) + \{ \tau_5^d (H_1^{d\dagger} \chi_2^d) (H_2^{d\dagger} \chi_1^d) + \text{h.c.} \} + \tau_6^d (|H_1^{d\dagger} \chi_2^d|^2 + |H_2^{d\dagger} \chi_1^d|^2) \\
& + \{ \tau_7^d \{ (H_2^{d\dagger} \chi_1^d) (\chi_2^{d\dagger} \chi_1^d) + (H_1^{d\dagger} \chi_2^d) (\chi_1^{d\dagger} \chi_2^d) \} + \text{h.c.} \} \\
& + \{ \omega_1^d \{ (H_s^{d\dagger} H_1^d) (H_2^{d\dagger} \chi_2^d) + (H_s^{d\dagger} H_2^d) (H_1^{d\dagger} \chi_1^d) \} + \text{h.c.} \} + \{ \omega_2^d \{ (H_s^{d\dagger} H_1^d) (\chi_1^{d\dagger} H_1^d) + (H_s^{d\dagger} H_2^d) (\chi_2^{d\dagger} H_2^d) \} + \text{h.c.} \} \\
& + \{ \omega_3^d \{ (H_s^{d\dagger} \chi_1^d) (H_1^{d\dagger} H_2^d) + (H_s^{d\dagger} \chi_2^d) (H_2^{d\dagger} H_1^d) \} + \text{h.c.} \}
\end{aligned} \tag{32}$$

This five Higgs potential is free from accidental symmetries. However, the combined potential  $V_u + V_d$  has an accidental  $SU(2) \times U(1) \times U(1)$  symmetry. It is broken explicitly by mixing terms, which couple the Higgs fields  $H_{s,1,2}^u$  and  $H_{s,1,2}^d/\chi_{1,2}^d$ .

The following potential  $V_{mixed}$  contains all such terms, which are invariant under the symmetry  $D_7 \times Z_2^{(aux)}$ :

$$\begin{aligned}
V_{mixed} = & \kappa_1 (H_s^{u\dagger} H_s^u) (H_s^{d\dagger} H_s^d) + \{\kappa_2 (H_s^{u\dagger} H_s^d)^2 + \text{h.c.}\} + \kappa_3 |H_s^{u\dagger} H_s^d|^2 \\
& + \kappa_4 \left( \sum_{i=1}^2 H_i^{u\dagger} H_i^u \right) \left( \sum_{i=1}^2 H_i^{d\dagger} H_i^d \right) + \tilde{\kappa}_4 \left( \sum_{i=1}^2 H_i^{u\dagger} H_i^u \right) \left( \sum_{i=1}^2 \chi_i^{d\dagger} \chi_i^d \right) \\
& + \{\kappa_5 \left( \sum_{i=1}^2 H_i^{u\dagger} H_i^d \right)^2 + \text{h.c.}\} + \kappa_6 |H_1^{u\dagger} H_1^d + H_2^{u\dagger} H_2^d|^2 \\
& + \kappa_7 (H_1^{u\dagger} H_1^u - H_2^{u\dagger} H_2^u) (H_1^{d\dagger} H_1^d - H_2^{d\dagger} H_2^d) + \tilde{\kappa}_7 (H_1^{u\dagger} H_1^u - H_2^{u\dagger} H_2^u) (\chi_1^{d\dagger} \chi_1^d - \chi_2^{d\dagger} \chi_2^d) \\
& + \{\kappa_8 (H_1^{u\dagger} H_1^d - H_2^{u\dagger} H_2^d)^2 + \text{h.c.}\} + \{\tilde{\kappa}_{[5-8]} (H_1^{u\dagger} \chi_1^d) (H_2^{u\dagger} \chi_2^d) + \text{h.c.}\} \\
& + \kappa_9 |H_1^{u\dagger} H_1^d - H_2^{u\dagger} H_2^d|^2 + \tilde{\kappa}_{[6+9]} (|H_1^{u\dagger} \chi_1^d|^2 + |H_2^{u\dagger} \chi_2^d|^2) + \kappa_{10} \{(H_2^{u\dagger} H_1^u) (H_1^{d\dagger} H_2^d) + \text{h.c.}\} \\
& + \{\kappa_{11} (H_2^{u\dagger} H_1^d) (H_1^{u\dagger} H_2^d) + \text{h.c.}\} + \{\tilde{\kappa}_{11} (H_1^{u\dagger} \chi_2^d) (H_2^{u\dagger} \chi_1^d) + \text{h.c.}\} + \kappa_{12} (|H_2^{u\dagger} H_1^d|^2 + |H_1^{u\dagger} H_2^d|^2) \\
& + \tilde{\kappa}_{12} (|H_1^{u\dagger} \chi_2^d|^2 + |H_2^{u\dagger} \chi_1^d|^2) + \kappa_{13} (H_s^{u\dagger} H_s^u) \left( \sum_{i=1}^2 H_i^{d\dagger} H_i^d \right) + \tilde{\kappa}_{13} (H_s^{u\dagger} H_s^u) \left( \sum_{i=1}^2 \chi_i^{d\dagger} \chi_i^d \right) \\
& + \{\kappa_{14} (H_s^{u\dagger} H_1^d) (H_s^{u\dagger} H_2^d) + \text{h.c.}\} + \{\tilde{\kappa}_{14} (H_s^{u\dagger} \chi_1^d) (H_s^{u\dagger} \chi_2^d) + \text{h.c.}\} + \kappa_{15} (|H_s^{u\dagger} H_1^d|^2 + |H_s^{u\dagger} H_2^d|^2) \\
& + \tilde{\kappa}_{15} (|H_s^{u\dagger} \chi_1^d|^2 + |H_s^{u\dagger} \chi_2^d|^2) + \kappa_{16} (H_s^{d\dagger} H_s^d) \left( \sum_{i=1}^2 H_i^{u\dagger} H_i^u \right) + \{\kappa_{17} (H_s^{d\dagger} H_1^u) (H_s^{d\dagger} H_2^u) + \text{h.c.}\} + \kappa_{18} \left( \sum_{i=1}^2 |H_s^{d\dagger} H_i^u|^2 \right) \\
& + \{\kappa_{19} (H_s^{u\dagger} H_s^d) \left( \sum_{i=1}^2 H_i^{u\dagger} H_i^d \right) + \text{h.c.}\} + \{\kappa_{20} (H_s^{u\dagger} H_s^d) \left( \sum_{i=1}^2 H_i^{d\dagger} H_i^u \right) + \text{h.c.}\} \\
& + \{\kappa_{21} \{(H_s^{u\dagger} H_1^u) (H_s^{d\dagger} H_2^d) + (H_s^{u\dagger} H_2^u) (H_s^{d\dagger} H_1^d)\} + \text{h.c.}\} + \{\kappa_{22} \{(H_s^{u\dagger} H_1^d) (H_s^{d\dagger} H_2^u) + (H_s^{u\dagger} H_2^d) (H_s^{d\dagger} H_1^u)\} + \text{h.c.}\} \\
& + \{\kappa_{23} \{(H_s^{u\dagger} H_1^u) (H_1^{d\dagger} H_s^d) + (H_s^{u\dagger} H_2^u) (H_2^{d\dagger} H_s^d)\} + \text{h.c.}\} + \{\kappa_{24} \{(H_s^{u\dagger} H_1^d) (H_1^{u\dagger} H_s^d) + (H_s^{u\dagger} H_2^d) (H_2^{u\dagger} H_s^d)\} + \text{h.c.}\} \\
& + \{\kappa_{25} \{(H_s^{d\dagger} H_1^u) (H_2^{u\dagger} \chi_2^d) + (H_s^{d\dagger} H_2^u) (H_1^{u\dagger} \chi_1^d)\} + \text{h.c.}\} + \{\kappa_{26} \{(H_s^{d\dagger} H_1^u) (\chi_1^{d\dagger} H_1^u) + (H_s^{d\dagger} H_2^u) (\chi_2^{d\dagger} H_2^u)\} + \text{h.c.}\} \\
& + \{\kappa_{27} \{(H_s^{d\dagger} \chi_1^d) (H_1^{u\dagger} H_2^d) + (H_s^{d\dagger} \chi_2^d) (H_2^{u\dagger} H_1^d)\} + \text{h.c.}\} + \{\kappa_{28} \{(H_s^{u\dagger} H_1^d) (H_2^{u\dagger} \chi_2^d) + (H_s^{u\dagger} H_2^d) (H_1^{u\dagger} \chi_1^d)\} + \text{h.c.}\} \\
& + \{\kappa_{29} \{(H_s^{u\dagger} H_1^d) (\chi_1^{d\dagger} H_1^u) + (H_s^{u\dagger} H_2^d) (\chi_2^{d\dagger} H_2^u)\} + \text{h.c.}\} + \{\kappa_{30} \{(H_s^{u\dagger} \chi_1^d) (H_1^{d\dagger} H_2^u) + (H_s^{u\dagger} \chi_2^d) (H_2^{d\dagger} H_1^u)\} + \text{h.c.}\} \\
& + \{\kappa_{31} \{(H_s^{u\dagger} \chi_1^d) (H_1^{u\dagger} H_2^d) + (H_s^{u\dagger} \chi_2^d) (H_2^{u\dagger} H_1^d)\} + \text{h.c.}\} + \{\kappa_{32} \{(H_s^{u\dagger} H_1^u) (H_2^{d\dagger} \chi_2^d) + (H_s^{u\dagger} H_2^u) (H_1^{d\dagger} \chi_1^d)\} + \text{h.c.}\} \\
& + \{\kappa_{33} \{(H_s^{u\dagger} H_1^u) (\chi_1^{d\dagger} H_1^d) + (H_s^{u\dagger} H_2^u) (\chi_2^{d\dagger} H_2^d)\} + \text{h.c.}\}
\end{aligned} \tag{33}$$

In our numerical analysis we restricted ourselves to the inclusion of a minimal number of terms from  $V_{mixed}$  which break all accidental symmetries such that only three Higgs particles remain massless which are eaten by the  $W^\pm$  and  $Z^0$  boson. As explained in the main part of the text, the three terms  $\kappa_2$ ,  $\kappa_5$  and  $\kappa_{19}$  are sufficient.

The numerical example in Section 4.1.1 and Section 4.2.1 needs the following VEV configuration

$$\begin{aligned}
\langle H_s^{d,u} \rangle &= 61.5 \text{ GeV} , \quad \langle H_1^d \rangle = \langle H_2^d \rangle = \langle \chi_1^d \rangle = \langle \chi_2^d \rangle = 61.5 \text{ GeV} , \quad \langle H_1^u \rangle = 61.5 e^{-\frac{3\pi i}{7}} \text{ GeV} \\
\text{and } \langle H_2^u \rangle &= 61.5 e^{\frac{3\pi i}{7}} \text{ GeV}
\end{aligned}$$

which allows real parameters in the potential  $V_d$ , as all fields  $H_s^d$ ,  $H_{1,2}^d$  and  $\chi_{1,2}^d$  have real VEVs. Furthermore we can remove the phase of  $\sigma_2^u$  such that we are left with three complex parameters stemming from  $V_{mixed}$ .

The mass parameters are at the electroweak scale:

$$\mu_s^u = 100 \text{ GeV} , \quad \mu_D^u = 200 \text{ GeV} , \quad \mu_s^d = 100 \text{ GeV} , \quad \mu_D^d = 200 \text{ GeV} \quad \text{and} \quad \tilde{\mu}_D^d = 150 \text{ GeV} .$$

One possible setup of parameters is then:

For  $V_u$  we take:

$$\lambda_s^u = 0.959337, \quad \lambda_1^u = 2.52548, \quad \lambda_2^u = 0.374967, \quad \lambda_3^u = -0.588842, \quad \sigma_1^u = 1.62353, \\ \sigma_2^u = -0.276964, \quad \sigma_3^u = -0.283914.$$

For  $V_d$  we set:

$$\lambda_s^d = 1.70438, \quad \lambda_1^d = 3.76598, \quad \tilde{\lambda}_1^d = 1.47549, \quad \lambda_2^d = -0.344036, \quad \tilde{\lambda}_2^d = -0.185157, \quad \lambda_3^d = -0.304589, \\ \tilde{\lambda}_3^d = -0.733236, \quad \sigma_1^d = 0.22429, \quad \tilde{\sigma}_1^d = 4.6792, \quad \sigma_2^d = -0.87457, \quad \tilde{\sigma}_2^d = -2.0284, \quad \sigma_3^d = 0.961454, \\ \tilde{\sigma}_3^d = 0.649984, \quad \tau_1^d = 2.96557, \quad \tau_2^d = 1.22903, \quad \tau_3^d = -2.02133, \quad \tau_4^d = -1.22242, \quad \tau_5^d = -2.31577, \\ \tau_6^d = 2.38236, \quad \tau_7^d = -0.660102, \quad \omega_1^d = 0.452165, \quad \omega_2^d = -2.112, \quad \omega_3^d = -1.63452.$$

and for the three complex nonzero couplings from  $V_{mixed}$ :

$$\kappa_2 = -0.638073 + i 0.0277608, \quad \kappa_5 = 0.312782 + i 0.140162, \quad \kappa_{19} = -0.278402 - i 0.124756.$$

Note that all parameters have absolute values smaller than 5 and hence they are still in the perturbative regime.

With these parameter values we obtain the desired VEV structure.

The Higgs masses are then

$$513 \text{ GeV}, \quad 499 \text{ GeV}, \quad 426 \text{ GeV}, \quad 414 \text{ GeV}, \quad 386 \text{ GeV}, \quad 365 \text{ GeV}, \quad 321 \text{ GeV}, \quad 266 \text{ GeV}, \quad 246 \text{ GeV}, \quad 227 \text{ GeV}, \\ 178 \text{ GeV}, \quad 159 \text{ GeV}, \quad 134 \text{ GeV}, \quad 81 \text{ GeV} \quad \text{and} \quad 55 \text{ GeV}$$

for the neutral scalars. Due to the explicit CP violation in the potential we can no longer distinguish between scalars and pseudo-scalars. For the charged scalar fields we get

$$367 \text{ GeV}, \quad 333 \text{ GeV}, \quad 294 \text{ GeV}, \quad 261 \text{ GeV}, \quad 145 \text{ GeV}, \quad 115 \text{ GeV} \quad \text{and} \quad 55 \text{ GeV}.$$

They are therefore in general too light to pass the constraints coming from direct searches as well as from bounds on FCNCs. Nevertheless, soft breaking terms of mass dimension two of the order of 10 TeV could lift the masses above these experimental bounds.

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